

Dark Matter Particle Scattering Processes and the Role of Sommerfeld Enhancement



UNIVERSITY OF TURIN
Department of Physics
Master's Degree in Theoretical Physics

Summer School GraSPA2026,
July 21, 2026

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Dark Matter

Dark Matter: a hypothetical new component of matter. Having suppressed (or null) electromagnetic interactions.

WIMP Hypothesis (*Weakly Interacting Massive Particle*): non-baryonic DM particles interacting only via weak interactions and having a sufficiently high mass ($m_x \geq \text{a few GeV}$).

Dark Matter Detection: Production in Colliders, Direct detection, *Indirect detection*: searching for the products of WIMP particle annihilation processes.

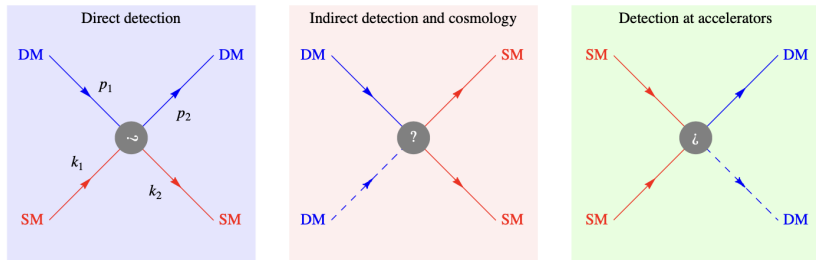


Figure 1: See Cirelli, Strumia, Zupan [3].

The Hulthén Potential: the cross section and S

An attractive Hulthén potential can be considered together with an approximation of the centrifugal term:

$$V_H(r) = -\frac{\alpha_x \delta e^{-\delta r}}{1 - e^{-\delta r}} \quad , \quad \tilde{V}_l(r) = \frac{l(l+1)}{m_x} \frac{\delta^2 e^{-\delta r}}{(1 - e^{-\delta r})^2}$$

with $\delta = km_\phi$, and solving the Schrödinger equation:

$$\left[\frac{d^2}{dr^2} + \frac{m_x^2 v^2}{4} + \frac{m_x \alpha_x \delta e^{-\delta r}}{1 - e^{-\delta r}} - \frac{l(l+1)\delta^2 e^{-\delta r}}{(1 - e^{-\delta r})^2} \right] \chi_l(r) = 0.$$

This leads to:

$$\chi_l(t) = \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l) \Gamma(\lambda_-^l)}{\Gamma(\omega) \Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right| t^{l+1} (1-t)^{iac} {}_2F_1(\lambda_+^l, \lambda_-^l, \omega, t)$$

This yields the phase shifts:

$$\delta_l = \arg \left(\frac{i \Gamma(\omega) \Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l) \Gamma(\lambda_-^l)} \right)$$

The Hulthén Potential: the cross section and S

and the cross-section:

$$\sigma_l = \frac{16\pi}{m_x^2 v^2} \sum_{l=0}^{\infty} (2l+1) \sin^2 \left\{ \arg \left(\frac{i\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)} \right) \right\}.$$

Sommerfeld Enhancement: a typical non-relativistic QM phenomenon involving an increase in the annihilation cross-section, compared to the perturbative calculation, between particles in the presence of an interaction potential. The Sommerfeld factor is:

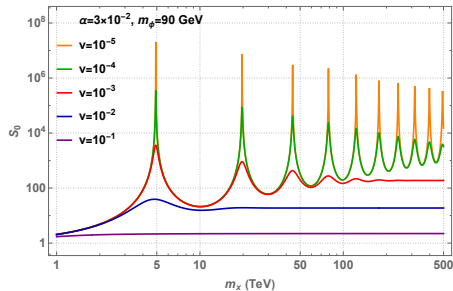
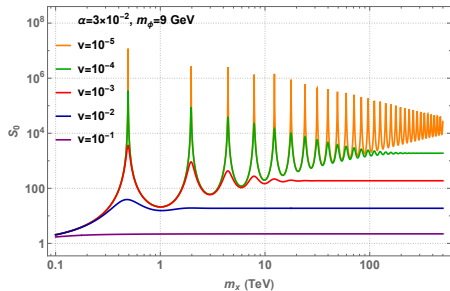
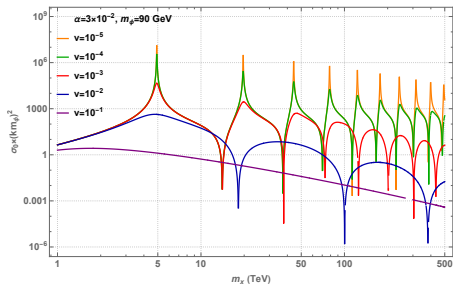
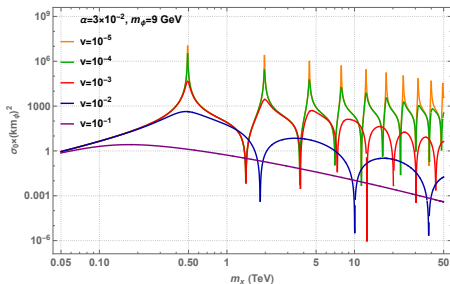
$$\begin{aligned} S_l &= \left| \frac{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}{\Gamma(1+l)\Gamma(1+l+2iac)} \right|^2 \\ &= \left| \frac{(1+iac+\sqrt{c-a^2c^2})_l (1+iac-\sqrt{c-a^2c^2})_l}{\Gamma(1+l)(1+2iac)_l} \right|^2 \times \\ &\quad \times \frac{\pi}{a} \frac{\sinh(2\pi ac)}{\cosh(2\pi ac) - \cosh(2\pi\sqrt{a^2c^2-c})} \end{aligned}$$

The Hulthén Potential: the cross section and S

The resonant values for S_l can be determined considering the low-velocity limit $v \rightarrow 0$ ($a \rightarrow 0$). Resonance occurs for:

$$m_x = \frac{\pi^2 m_\phi}{6\alpha_x} n^2 \quad , \quad n \in \mathbb{N}.$$

The Hulthén Potential: plots of S factor and σ for $l=0$



Conclusions

- The fundamental nature of Dark Matter remains one of the most significant open questions in contemporary physics.
- WIMP particles are among the leading candidates for DM.
- Sommerfeld enhancement could play a crucial role in the detection of indirect signals from WIMPs.

Thank You

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Indications of Dark Matter

Dark Matter: a hypothetical new component of matter. Having suppressed (or null) electromagnetic interactions, its observational detection is highly complex.

Experimental evidence:

- Bullet Cluster
- Galaxy rotation curves
- Gravitational lensing of galaxy clusters
- Cosmic microwave background

Indications of Dark Matter: Bullet Cluster

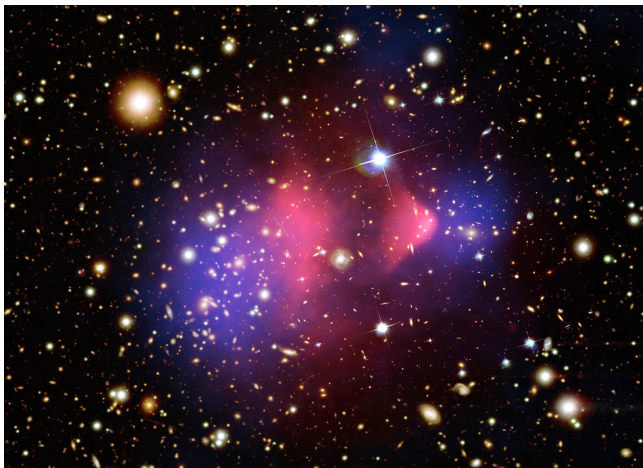


Figure 2: The Bullet Cluster. The image shows the hot gas (in pink) traced via X-rays, galaxies and stars (the yellow and white areas), and the mass distribution (in blue). Credit: X-ray: NASA/CXC/CfA/M.Markevitch et al.; Optical: NASA/STScI; Magellan/U.Arizona/D.Clowe et al.; Lensing Map: NASA/STScI; ESO WFI; Magellan/U.Arizona/D.Clowe et al.

Indications of Dark Matter: Rotation Curves

Experimentally we observe:

$$M(r) \sim M \quad r \sim \mathcal{O}(10) \text{ kpc.}$$

From Newton's equation:

$$G \frac{mM(r)}{r^2} = m \frac{v_c^2}{r}$$
$$v_c = \sqrt{\frac{GM(r)}{r}}.$$

Therefore, one must assume the existence of a DM component such that $\rho(r) \propto 1/r^2$ for $r \sim \mathcal{O}(10)$ kpc.

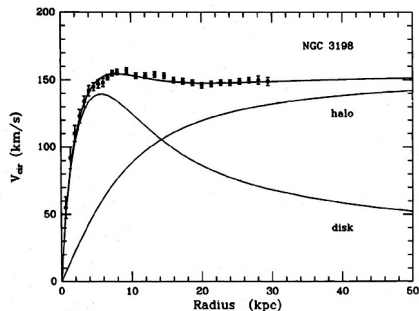


Figure 3: Reference: [link](#). Rotation curve of the spiral galaxy **NGC 3198** (located approximately 47 million light-years from Earth).

WIMP Hypothesis and Scattering Processes of DM Particles

WIMP Hypothesis (Weakly Interacting Massive Particle): non-baryonic DM particles interacting only via weak interactions and having a sufficiently high mass ($m_x \geq$ a few GeV).

Dark Matter Detection:

- Production in Colliders
- Direct detection
- Indirect detection: searching for the products of WIMP particle annihilation processes.

Sommerfeld Enhancement: a typical non-relativistic QM phenomenon involving an increase in the annihilation cross-section, compared to the perturbative calculation, between particles in the presence of an interaction potential.

The Sommerfeld Enhancement: $e^+ + e^- \rightarrow 2\gamma$ Annihilation

Annihilation Hamiltonian:

$$H_{ann} = U_0 \delta^3(\mathbf{r}).$$

Annihilation cross-section:

$$\sigma_0 = |\psi_0(0)|^2.$$

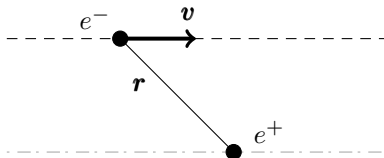


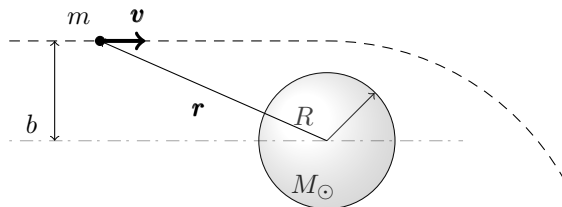
Figure 4: Graphical representation of the $e^+ + e^- \rightarrow 2\gamma$ annihilation experiment.

Introducing the interaction potential $V(r)$, we have:

$$\sigma = |\psi(0)|^2 \quad \Longrightarrow \quad S = \frac{\sigma}{\sigma_0} = \frac{|\psi(0)|^2}{|\psi_0(0)|^2}.$$

The Sommerfeld Enhancement: Classical Analogy

A planet of mass m interacts gravitationally with the Sun $M_{\odot}$.



Ignoring gravity:

$$\sigma_0 = \pi R^2.$$

Considering the gravitational interaction $V(r) = -G \frac{m M_{\odot}}{r}$:

$$\sigma = \pi R^2 \left(1 + \frac{v_{esc}^2}{v^2} \right) \quad \Rightarrow \quad S = \frac{\sigma}{\sigma_0} = 1 + \frac{v_{esc}^2}{v^2}.$$

The Hulthén Potential

An attractive Hulthén potential can be considered together with an approximation of the centrifugal term:

$$V_H(r) = -\frac{\alpha_x \delta e^{-\delta r}}{1 - e^{-\delta r}} \quad , \quad \tilde{V}_l(r) = \frac{l(l+1)}{m_x} \frac{\delta^2 e^{-\delta r}}{(1 - e^{-\delta r})^2}$$

and solving the Schrödinger equation:

$$\left(-\frac{\hbar^2}{2\mu} \Delta + V(\mathbf{r}) \right) \Psi(\mathbf{r}) = E \Psi(\mathbf{r}) \implies \left(\frac{d^2}{dr^2} + \kappa^2 - 2\mu V_{eff.}(r) \right) \chi_l(r) = 0$$

$$\left[\frac{d^2}{dr^2} + \frac{m_x^2 v^2}{4} + \frac{m_x \alpha_x \delta e^{-\delta r}}{1 - e^{-\delta r}} - \frac{l(l+1) \delta^2 e^{-\delta r}}{(1 - e^{-\delta r})^2} \right] \chi_l(r) = 0.$$

where it is convenient to define:

$$a = \frac{v}{2\alpha_x} \quad , \quad c = \frac{\alpha_x m_x}{\delta} \quad \delta = \frac{\pi^2}{6} m_\phi.$$

Hulthén potential: Schrödinger Equation

Indeed, with an attractive Hulthén potential can be considered:

$$V_H(r) = -\frac{\alpha_x \delta e^{-\delta r}}{1 - e^{-\delta r}}$$

and the approximation of the centrifugal term:

$$\tilde{V}_l(r) = \frac{l(l+1)}{m_x} \frac{\delta^2 e^{-\delta r}}{(1 - e^{-\delta r})^2}.$$

To solve the Schrödinger equation, one can define:

$$x = \alpha_x m_x r \quad , \quad a = \frac{v}{2\alpha_x} \quad , \quad c = \frac{\alpha_x m_x}{\delta}$$

so that:

$$\left(-\frac{\hbar^2}{2\mu} \Delta + V(\mathbf{r}) \right) \Psi(\mathbf{r}) = E \Psi(\mathbf{r}) \implies \left(\frac{d^2}{dr^2} + \kappa^2 - 2\mu V_{eff.}(r) \right) \chi_l(r) = 0$$

Hulthén potential: Schrödinger Equation

$$\left[\frac{d^2}{dx^2} + a^2 + \frac{1}{c} \frac{e^{-x/c}}{1 - e^{-x/c}} - \frac{l(l+1)}{c^2} \frac{e^{-x/c}}{(1 - e^{-x/c})^2} \right] \chi_l(r) = 0.$$

By changing the variable to:

$$t = 1 - e^{-x/c}$$

and reparameterizing the wavefunction:

$$\chi_l(x) = t^{l+1} (1-t)^{iac} f(t)$$

we obtain:

$$\left\{ t(1-t) \frac{d^2}{dt^2} + \left[2 + 2l - (2 + 2l + 2iac + 1)t \right] \frac{d}{dt} - \left[l(l+1) + (l+1)(2iac + 1) - c \right] \right\} f(t) = 0$$

Hulthén potential: Schrödinger Equation

where we can define:

$$\begin{cases} \lambda_{\pm}^l = 1 + l + iac \pm \sqrt{c - a^2 c^2} \\ \omega = 2 + 2l \end{cases}$$

such that:

$$\left\{ t(1-t) \frac{d^2}{dt^2} + \left[\omega - (\lambda_+^l + \lambda_-^l + 1)t \right] \frac{d}{dt} - \lambda_+^l \lambda_-^l \right\} f(t) = 0.$$

Thus, the solution is:

$$\chi_l(t) = A t^{l+1} (1-t)^{iac} {}_2F_1(\lambda_+^l, \lambda_-^l, \omega, t)$$

The constant A can be fixed by requiring that the part of $\chi_l(x)$ behaving as an outgoing plane wave $e^{i\mathbf{k} \cdot \mathbf{r}}$ for $r \rightarrow \infty$ has a modulus of 1.

Hulthén potential: Wavefunction Normalization

The wavefunction is:

$$\chi_l(t) = A t^{l+1} (1-t)^{iac} {}_2F_1(\lambda_+^l, \lambda_-^l, \omega, t)$$

The constant A can be fixed by requiring that the part of $\chi_l(x)$ behaving as an outgoing plane wave $e^{i\mathbf{k}\cdot\mathbf{r}}$ for $r \rightarrow \infty$ has a modulus of 1.

Using one of the 5 Bolza formulas for the Hypergeometric function, along with ${}_2F_1(a, b, c, z) \rightarrow 1$ as $z \rightarrow 0$, we obtain the behavior for $t \rightarrow 1$ ($r \rightarrow \infty$):

$$\begin{aligned} \chi_l(t) = A t^{l+1} (1-t)^{iac} & \left\{ \frac{\Gamma(\omega)\Gamma(\omega - \lambda_+^l - \lambda_-^l)}{\Gamma(\omega - \lambda_+^l)\Gamma(\omega - \lambda_-^l)} \times \right. \\ & \times {}_2F_1(\lambda_+^l, \lambda_-^l, \lambda_+^l + \lambda_-^l - \omega + 1, 1-t) + \\ & + \frac{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)} (1-t)^{\omega - \lambda_+^l - \lambda_-^l} \times \\ & \left. \times {}_2F_1(\omega - \lambda_+^l, \omega - \lambda_-^l, 1 + \omega - \lambda_+^l - \lambda_-^l, 1-t) \right\}. \end{aligned}$$

Hulthén potential: Wavefunction Normalization

Note that:

$$\Gamma(\omega - \lambda_+^l - \lambda_-^l) = \left(\Gamma(\lambda_+^l + \lambda_-^l - \omega) \right)^*$$

$$\Gamma(\omega - \lambda_+^l) = \left(\Gamma(\lambda_-^l) \right)^*$$

$$\Gamma(\omega - \lambda_-^l) = \left(\Gamma(\lambda_+^l) \right)^*$$

to rewrite the wavefunction as:

$$\begin{aligned} \chi_l(t) = & A \left(\frac{\Gamma(\omega) \Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l) \Gamma(\lambda_-^l)} \right)^* t^{l+1} (1-t)^{iac} \times \\ & \times {}_2F_1(\lambda_+^l, \lambda_-^l, \lambda_+^l + \lambda_-^l - \omega + 1, 1-t) + A \left(\frac{\Gamma(\omega) \Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l) \Gamma(\lambda_-^l)} \right) \times \\ & \times t^{l+1} (1-t)^{-iac} {}_2F_1(\omega - \lambda_+^l, \omega - \lambda_-^l, 1 + \omega - \lambda_+^l - \lambda_-^l, 1-t). \end{aligned}$$

Hulthén potential: Wavefunction Normalization

Taking the limit $t \rightarrow 1$ gives:

$$t^{l+1} \rightarrow 1 \quad , \quad (1-t)^{i\alpha} \rightarrow (e^{-x/c})^{i\alpha}$$

and the wavefunction becomes:

$$\begin{aligned} \chi_l(x) \sim & A \left(\frac{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)} \right)^* e^{-i\alpha x} + \\ & + A \left(\frac{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)} \right) e^{i\alpha x} \end{aligned}$$

Hulthén potential: Wavefunction Normalization

$$\begin{aligned}\chi_l(x) \sim A & \left| \frac{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)} \right| \times \\ & \times \left[e^{iax} \exp \left\{ i \arg \left(\frac{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)} \right) \right\} + \right. \\ & \left. + e^{-iax} \exp \left\{ -i \arg \left(\frac{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)} \right) \right\} \right].\end{aligned}$$

The constant A is determined as:

$$A = \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right|.$$

The asymptotic behavior of $\chi_l(x)$ can be rewritten as:

$$\chi_l(x) \sim \cos \left\{ ax + \arg \left(\frac{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)} \right) \right\}$$

Hulthén potential: Wavefunction Normalization

using $\cos \alpha = \sin \left(\alpha + \frac{\pi}{2} \right)$ and $\arg(zw) = \arg z + \arg w$, it becomes:

$$\chi_l(x) \underset{x \rightarrow \infty}{\sim} \sin \left\{ ax + \arg \left(\frac{i\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)} \right) \right\}.$$

Where we can notice $ax = kr$.

Hulthén potential: The cross section

This leads to:

$$\chi_l(t) = \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l) \Gamma(\lambda_-^l)}{\Gamma(\omega) \Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right| t^{l+1} (1-t)^{iac} {}_2F_1(\lambda_+^l, \lambda_-^l, \omega, t)$$
$$\chi_l(x) \underset{x \rightarrow \infty}{\sim} \sin \left\{ kr + \arg \left(\frac{i \Gamma(\omega) \Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l) \Gamma(\lambda_-^l)} \right) \right\}.$$

Knowing the expected behavior from the theory:

$$\chi_l(r) \underset{r \rightarrow \infty}{\sim} \sin \left(kr - \frac{\pi}{2} l + \delta_l \right)$$

one can determine δ_l by computing:

$$\phi_l = \arg \left(\frac{i \Gamma(\omega) \Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l) \Gamma(\lambda_-^l)} \right) - \arg \left(\frac{i \Gamma(\omega) \Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l) \Gamma(\lambda_-^l)} \right) \Big|_{\alpha_x \rightarrow 0}.$$

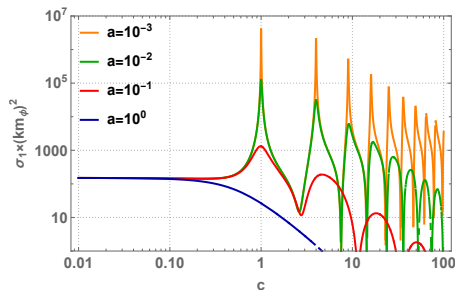
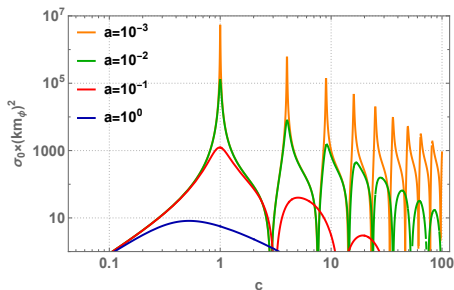
Hulthén potential: The cross section

This yields the phase shifts:

$$\delta_l = \arg \left(\frac{i\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)} \right)$$

and the cross-section:

$$\sigma_l = \frac{16\pi}{m_x^2 v^2} \sum_{l=0}^{\infty} (2l+1) \sin^2 \left\{ \arg \left(\frac{i\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)} \right) \right\}.$$



Hulthén potential: Wavefunction Behavior as $x \rightarrow 0$

Starting from:

$$\chi_l(t) = \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l) \Gamma(\lambda_-^l)}{\Gamma(\omega) \Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right| t^{l+1} (1-t)^{iac} {}_2F_1(\lambda_+^l, \lambda_-^l, \omega, t)$$

we can expand it as $x \rightarrow 0$, i.e., as $t = 1 - e^{-x/c} \rightarrow 0$, to find:

$$\chi_l(r) \rightarrow B_l \frac{(\kappa r)^{l+1}}{(2l+1)!!}$$

and specifically the coefficient B_l , which represents the distortion introduced by the potential $V_H(r)$.

Hulthén potential: Wavefunction Behavior as $x \rightarrow 0$

We have:

$$\begin{aligned}\chi_l(x) &\underset{x \rightarrow 0}{\sim} \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l) \Gamma(\lambda_-^l)}{\Gamma(\omega) \Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right| (1 - e^{-x/c})^{l+1} \cdot 1 \\ &= \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l) \Gamma(\lambda_-^l)}{\Gamma(\omega) \Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right| \left(1 - \left(1 - \frac{x}{c} + \dots \right) \right)^{l+1} \\ &= \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l) \Gamma(\lambda_-^l)}{\Gamma(\omega) \Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right| \left(\frac{x}{c} \right)^{l+1}\end{aligned}$$

with the defined parameters, we have $\frac{x}{c} = km_\phi r$, where $k = \pi^2/6$, so:

$$\chi_l(x) \underset{x \rightarrow 0}{\sim} (km_\phi r)^{l+1} \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l) \Gamma(\lambda_-^l)}{\Gamma(\omega) \Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right|.$$

Hulthén potential: Wavefunction Behavior as $x \rightarrow 0$

Using the parameter definitions again, it can be seen that:

$$\kappa r = \frac{m_x}{2} v r \quad , \quad k m_\phi r = k m_\phi \frac{m_x v}{2} \frac{2}{m_x v} r = \frac{2 k m_\phi}{m_x v} \kappa r = \frac{\kappa r}{ac}$$

allowing us to rewrite:

$$\begin{aligned} \chi_l(x) &\sim \frac{(k m_\phi)^{l+1}}{(\kappa)^{l+1}} (\kappa r)^{l+1} \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l) \Gamma(\lambda_-^l)}{\Gamma(\omega) \Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right| \\ &\sim \frac{(2l+1)!!}{(ac)^{l+1}} \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l) \Gamma(\lambda_-^l)}{\Gamma(\omega) \Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right| \frac{(\kappa r)^{l+1}}{(2l+1)!!} \end{aligned}$$

Comparing:

$$\begin{aligned} \chi_l(x) &\sim \frac{(2l+1)!!}{(ac)^{l+1}} \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l) \Gamma(\lambda_-^l)}{\Gamma(\omega) \Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right| \frac{(\kappa r)^{l+1}}{(2l+1)!!} \\ \chi_l(r) &\rightarrow B_l \frac{(\kappa r)^{l+1}}{(2l+1)!!} \end{aligned}$$

Hulthén potential: Wavefunction Behavior as $x \rightarrow 0$

it follows that:

$$B_l = \frac{(2l+1)!!}{(ac)^{l+1}} \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l) \Gamma(\lambda_-^l)}{\Gamma(\omega) \Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right|.$$

To determine the Sommerfeld factor:

$$S = \frac{|B_l|^2}{|B_l(c \rightarrow 0, ac = u)|^2}$$

we need to compute $B_l(c \rightarrow 0, ac = u) = B_l(V \rightarrow 0)$, where we have set $ac = \frac{m_x v}{2k m_\phi} = u$. We have:

$$\begin{aligned} B_l &= \frac{(2l+1)!!}{(ac)^{l+1}} \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l) \Gamma(\lambda_-^l)}{\Gamma(\omega) \Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right| \\ &= \frac{(2l+1)!!}{2u^{l+1}} \left| \frac{\Gamma(1+l+iu+ac\sqrt{\frac{1}{a^2c}-1}) \Gamma(1+l+iu-ac\sqrt{\frac{1}{a^2c}-1})}{\Gamma(2l+2) \Gamma(2iu)} \right| \end{aligned}$$

Hulthén potential: Wavefunction Behavior as $x \rightarrow 0$

taking the limit $c \rightarrow 0$ and $a \rightarrow \infty$:

$$B_l = \frac{(2l+1)!!}{2u^{l+1}} \left| \frac{\Gamma(1+l+iu+iu)\Gamma(1+l+iu-iu)}{\Gamma(2l+2)\Gamma(2iu)} \right|$$

which ultimately yields:

$$B_l(c \rightarrow 0, ac = u) = \frac{(2l+1)!!}{2(ac)^{l+1}} \left| \frac{\Gamma(1+l+2iac)\Gamma(1+l)}{\Gamma(\omega)\Gamma(2iac)} \right|.$$

Hulthén potential: the Sommerfeld factor

Having determined the two distortion coefficients:

$$B_l = \frac{(2l+1)!!}{(ac)^{l+1}} \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l) \Gamma(\lambda_-^l)}{\Gamma(\omega) \Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right|$$
$$B_l(c \rightarrow 0, ac = u) = \frac{(2l+1)!!}{2(ac)^{l+1}} \left| \frac{\Gamma(1+l+2iac) \Gamma(1+l)}{\Gamma(\omega) \Gamma(2iac)} \right|$$

the Sommerfeld factor results in:

$$S_l = \frac{|B_l|^2}{|B_l(c \rightarrow 0, ac = u)|^2}$$
$$= \left(\frac{(2l+1)!!}{2(ac)^{l+1}} \left| \frac{\Gamma(\lambda_+^l) \Gamma(\lambda_-^l)}{\Gamma(\omega) \Gamma(2iac)} \right| \right)^2 \left(\frac{2(ac)^{l+1}}{(2l+1)!!} \left| \frac{\Gamma(\omega) \Gamma(2iac)}{\Gamma(1+l+2iac) \Gamma(1+l)} \right| \right)^2$$
$$= \left| \frac{\Gamma(\lambda_+^l) \Gamma(\lambda_-^l)}{\Gamma(1+l) \Gamma(1+l+2iac)} \right|^2.$$

Hulthén potential: the Sommerfeld factor

which, by utilizing:

$$|\Gamma(i\alpha)|^2 = \frac{\pi}{\alpha \sinh(\pi\alpha)} \quad , \quad |\Gamma(1+i\alpha)|^2 = \frac{\pi\alpha}{\sinh(\pi\alpha)} \quad , \quad \alpha \in \mathbb{R}$$

can be rewritten as:

$$S_l = \left| \frac{(1+iac + \sqrt{c-a^2c^2})_l (1+iac - \sqrt{c-a^2c^2})_l}{\Gamma(1+l)(1+2iac)_l} \right|^2 \times \\ \times \left| \frac{\Gamma(1+iac + i\sqrt{a^2c^2-c})\Gamma(1+iac - i\sqrt{a^2c^2-c})}{\Gamma(1+2iac)} \right|^2 .$$

$$S_l = \left| \frac{(1+iac + \sqrt{c-a^2c^2})_l (1+iac - \sqrt{c-a^2c^2})_l}{\Gamma(1+l)(1+2iac)_l} \right|^2 \times \\ \times \frac{\pi(ac + \sqrt{a^2c^2-c})\pi(ac - \sqrt{a^2c^2-c})}{\sinh(\pi(ac + \sqrt{a^2c^2-c}))\sinh(\pi(ac - \sqrt{a^2c^2-c}))} \cdot \frac{\sinh(2\pi ac)}{\pi(2ac)} .$$

Hulthén potential: the Sommerfeld factor

We can make use of:

$$\sinh x \sinh y = \frac{1}{2} (\cosh(x+y) - \cosh(x-y))$$

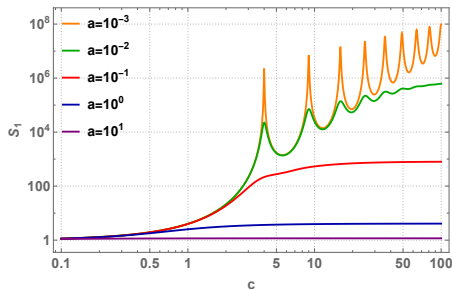
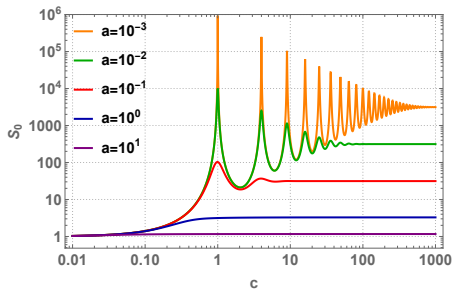
to obtain the final result:

$$S_l = \left| \frac{(1 + iac + \sqrt{c - a^2 c^2})_l (1 + iac - \sqrt{c - a^2 c^2})_l}{\Gamma(1+l)(1+2iac)_l} \right|^2 \times \\ \times \frac{\pi}{a} \frac{\sinh(2\pi ac)}{\cosh(2\pi ac) - \cosh(2\pi \sqrt{a^2 c^2 - c})}.$$

The cases for $l=0$ and $l=1$ are:

$$S_0 = \frac{\pi}{a} \frac{\sinh(2\pi ac)}{\cosh(2\pi ac) - \cosh(2\pi \sqrt{a^2 c^2 - c})} \quad , \quad S_1 = \frac{(1-c)^2 + 4a^2 c^2}{1 + 4a^2 c^2} S_0$$

Hulthén potential: the Sommerfeld factor



The resonant values for S_l can be determined:

$$S_l = \left| \frac{(1 + iac + \sqrt{c - a^2 c^2})_l (1 + iac - \sqrt{c - a^2 c^2})_l}{\Gamma(1 + l)(1 + 2iac)_l} \right|^2 \times$$

$$\times \frac{\pi}{a} \frac{\sinh(2\pi ac)}{\cosh(2\pi ac) - \cosh(2\pi \sqrt{a^2 c^2 - c})}$$

Hulthén potential: the Sommerfeld factor

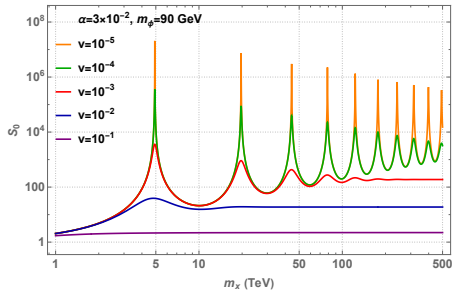
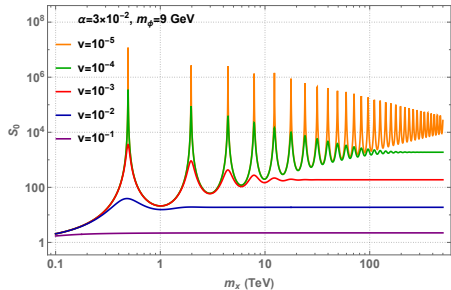
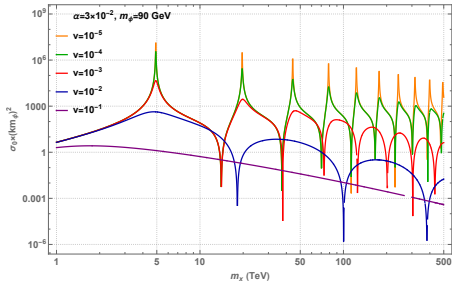
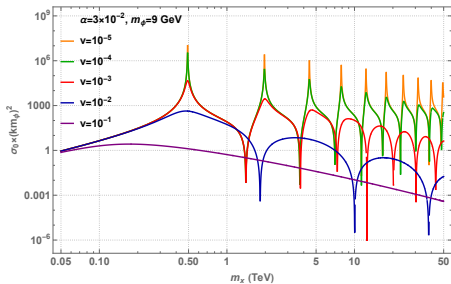
considering the low-velocity limit $v \rightarrow 0$ ($a \rightarrow 0$). Resonance occurs for:

$$m_x = \frac{\pi^2 m_\phi}{6\alpha_x} n^2, \quad n \in \mathbb{N}.$$

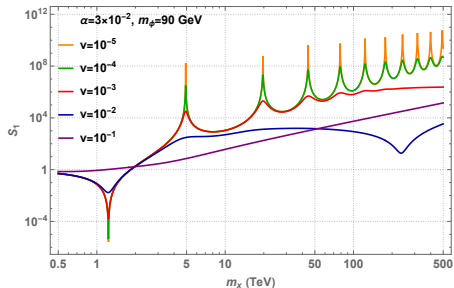
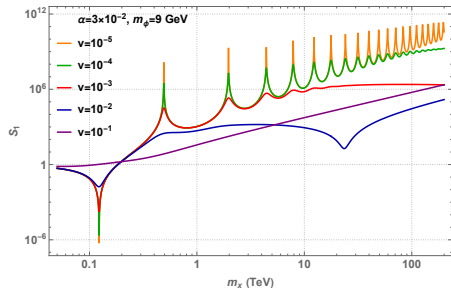
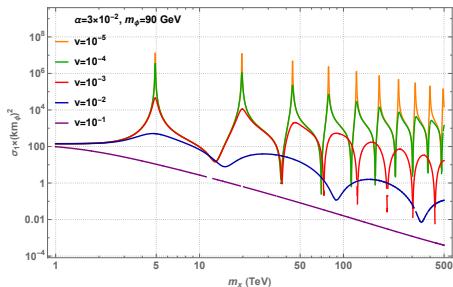
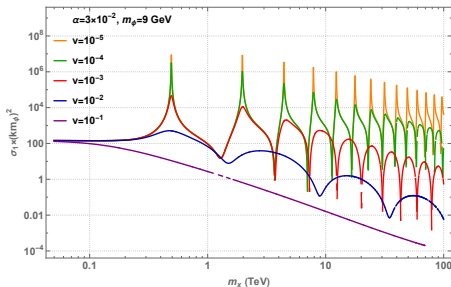
around which the behavior of S is:

$$S = \frac{4\alpha_x k m_\phi}{m_x v^2}.$$

Hulthén potential: plots of the S factor and σ for $l=0$



Hulthén potential: plots of the S factor and σ for $l = 1$



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