Different integral parametrizations: Baikov's method and Mellin-Barnes representation

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Abstract

In the introductory section §1 we discuss encountered problem we have to face when calculating certain Feynman diagrams and how we addressed these issues during the MSc course: *Advanced Quantum Field Theory*.

Subsequently, we analyze some alternative parametrizations of Feynman integrals. In the section §2 we will analyze the *Melli-Barnes representation* method, going through some examples to obtain well-known results. The Appendix A contains all the well-known results given by Smirnov [2] or found in the MSc course.

Throughout the document, several integrals are computed via `Mathematica`. The corresponding notebook can be found in the dedicated folder of the linked Github repository.

The main reference for this project is Smirnov [2], but other well-known textbooks [1, 3, 4] have also been very useful. Finally, the lecture notes of the MSc course were essential.

Contents

1 Introduction

As we saw during the MSc course Advanced Quantum Field Theory, several issues arise when evaluating certain Feynman diagrams. Specifically, we observed that divergences emerge when integrating Lorentz-covariant Feynman integrals over loop momenta. Over the years, a great variety of methods has been developed to evaluate these integrals. Naturally, some of these methods are more effective than others, depending on the specific problem at hand and the required degree of analytical or numerical complexity.

During the MSc course, we analyzed Feynman-Schwinger parameters (α-representation), Symanzik polynomials, and the Passarino-Veltman tensor reduction to evaluate loop momentum integrals. However, in this term project, we will focus on the Mellin-Barnes representation and the reduction to master integrals via Baikov's method. We will present some one-loop calculations and provide general recipes for evaluating multi-loop problems.

In these notes, we will use the Fourier transform (FT) of the propagators:

$$\tilde{D}_{F,i}(p) \equiv \int \mathrm{d}^4x\, e^{ip\cdot x}\, D_{F,i}(x) = \frac{iZ_i(p)}{(p^2 - m_i^2 + i\emptyset^+)^{a_i}} \tag{1}$$

where m_i is the corresponding mass, Z_i is a polynomial, and $a_i \in \{1, 2\}$ are the propagator indices, introduced to ensure generality. For a scalar field propagator, we have $Z = 1$ and $a = 1$, yielding:

$$\tilde{D}_{F,i}(p) = \frac{i}{p^2 - m_i^2 + i\emptyset^+}.$$

We will usually omit the causal $i\emptyset^+$ for brevity. We also omit the factors of i and $(2\pi)^4$ that traditionally enter into standard Feynman rules (in particular, in the definition of the FT); these can be restored at the end of the calculation.

During the MSc course, we analyzed the Feynman and Schwinger parametrizations (which are called the alpha representation by Smirnov [2]):

$$\frac{1}{(-k^2 + m^2)^\lambda} = \frac{i^\lambda}{\Gamma(\lambda)} \int_0^\infty \mathrm{d}\alpha, \alpha^{\lambda-1}, e^{i(k^2-m^2)\alpha} \tag{2}$$

Although we won't employ this regularization method directly in these notes, we will utilize some of its results, which are collected in Appendix A.

2 Mellin-Barnes representation

One althernative to Feynman parameter, for example, when dealing with Feynman integrals are the *Mellin integrals*. These are integrals over contours in a complex plane along the immaginary axis of a product and ratio of gamma function.

The key ingredient of the method presented in this chapter is the MB representation used to replace a sum of two terms raised to some power by the product of these terms raised to some powers. Our goal is to use such a factorization in order to achieve the possibility to perform integrations in terms of gamma functions, at the cost of introducing extra Mellin integrations. Then one obtains a multiple Mellin integral of gamma functions in the numerator and denominator. The next step is the resolution of the singularities in ϵ by means of shifting contours and taking residues. It turns out that multiple MB integrals are very convenient for this purpose. The final step is to perform at least some of the Mellin integrations explicitly, by means of the first and the second Barnes lemma and their corollaries and/or evaluate these integrals by closing the integration contours in the complex plane and summing up corresponding series.

Usually, one applies it starting from alpha or Feynman parametric integrals.

2.1 One-loop example

The starting point, and the basic tool is the formula:

$$\frac{1}{(X+Y)^{\lambda}} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}z\,\Gamma(\lambda+z)\,\Gamma(-z)\,\frac{Y^z}{X^{\lambda+z}}. \tag{3}$$

The contour of integration is chosen in the standard way: we can notice that the poles with a $\Gamma(\lambda+z)$ dependence (or in general $\Gamma(\cdots+z)$ dependence), so $-\lambda+n$ with $n\in\mathbb{N}$, are to the left of the contour (we can call them for brevity[1] the *left poles*) and the poles with a $\Gamma(-z)$ dependence (or in general $\Gamma(\cdots-z)$ dependence), so the posivite integer (the *right poles*) are to the right of it. You can see the figure 1.

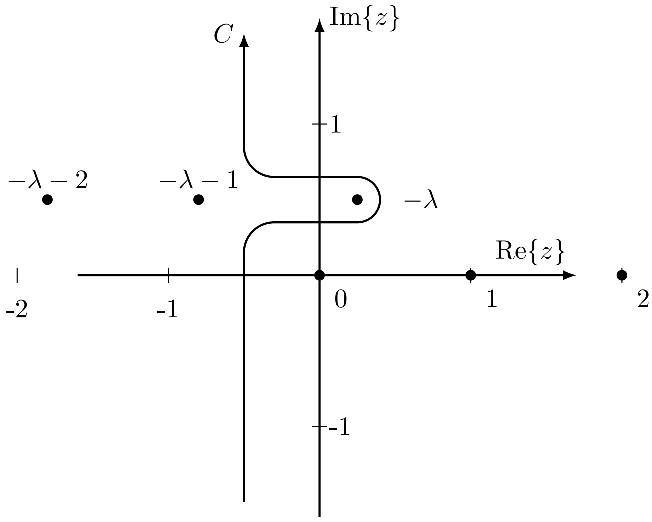


Figure 1: Possible integration contour in (3) for $\lambda=-\frac{1}{4}-\frac{i}{2}$.

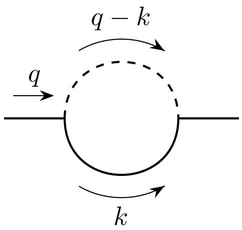
In general we shall use de decomposition $X+Y$ of various functions in integrals over Feynman (or alpha) parameters. But a more useful way to procede is to write the massive propagator (so $1/(m^2-k^2)$), which can be complicated at multi-loop calculation, in terms of massless ones (so in terms of $1/(-k^2)$). So putting $X=-k^2$ and $Y=m^2$ we have:

$$\frac{1}{(m^2-k^2)^{\lambda}} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}z\,\Gamma(\lambda+z)\,\Gamma(-z)\,\frac{(m^2)^z}{(-k^2)^{\lambda+z}} \tag{4}$$

[1] This terminology is useful and, although it often happens that the first right pole is to the left of the first left pole of a given integrand, this, hopefully, will not cause misunderstanding.

so we have the massless propagator with a simple kinematic term m^2 (to some power), but we gain an integral in the immaginary plane and some gamma function.

Example 1. *One-loop propagator.* We have the Feynman diagram:



$$\tag{5}$$

where the dashed line denotes a massless propagator and the solid one a massive propagator. Applying the Feynman rules we get the Feynman integral:

$$I_2^{(1)[d]}(q^2, m^2; a_1, a_2, d) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^{a_1}[(q-k)^2]^{a_2}} \tag{6}$$

with general powers of the two propagator; if we insert (4), with $\lambda = a_1$, into it we get:

$$\begin{aligned} I_2^{(1)[d]}(q^2, m^2; a_1, a_2, d) &= \int \frac{\mathrm{d}^d k}{[(q-k)^2]^{a_2}} \frac{1}{\Gamma(a_1)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z\, \Gamma(a_1+z)\, \Gamma(-z)\, \frac{(m^2)^z}{(-k^2)^{a_1+z}} &(7)\\ &= \frac{(-1)^{a_1}}{\Gamma(a_1)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z\, \Gamma(a_1+z)\, \Gamma(-z)\, (m^2)^z \int \frac{\mathrm{d}^d k}{(-k^2)^{a_1+z}[(q-k)^2]^{a_2}} &(8)\end{aligned}$$

where the term $(-1)^{a_1}$ is due to the change of the sign in applying (4). Now we can use the general relation (that we demonstrate in Appendix A):

$$\int \frac{\mathrm{d}^d k}{(-k^2)^{\lambda_1}[-(q-k)^2]^{\lambda_2}} = i\pi^{d/2} \frac{\Gamma(\lambda_1+\lambda_2+\epsilon-2)\Gamma(2-\epsilon-\lambda_1)\Gamma(2-\epsilon-\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(4-\lambda_1-\lambda_2-2\epsilon)} \frac{1}{(-q^2)^{\lambda_1+\lambda_2+\epsilon-2}} \tag{9}$$

because in this way, putting $\lambda_1 = a_1 + z$ and $\lambda_2 = a_2$ we get:

$$\begin{aligned} I_2^{(1)[d]}(q^2, m^2; a_1, a_2, d) &= \frac{(-1)^{a_1}}{\Gamma(a_1)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z\, \Gamma(a_1+z)\, \Gamma(-z)\, (m^2)^z \times \\ &\quad \times \frac{i\pi^{d/2}(-1)^{a_2}}{(-q^2)^{a_1+a_2+z+\epsilon-2}} \frac{\Gamma(a_1+a_2+z+\epsilon-2)\Gamma(2-\epsilon-a_1-z)\Gamma(2-\epsilon-a_2)}{\Gamma(a_1+z)\Gamma(a_2)\Gamma(4-a_1-a_2-z-2\epsilon)} &(10)\\ &= \frac{i\pi^{d/2}(-1)^{a_1+a_2}\,\Gamma(2-\epsilon-a_2)}{\Gamma(a_1)\Gamma(a_2)(-q^2)^{a_1+a_2+\epsilon-2}} \times \\ &\quad \times \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \Gamma(a_1+a_2+\epsilon-2+z) \frac{\Gamma(2-\epsilon-a_1-z)\Gamma(-z)}{\Gamma(4-a_1-a_2-2\epsilon-z)} &(11)\end{aligned}$$

where the term $(-1)^{a_2}$ is due to the change of the sign in $[(q-k)^2]^{a_2}$. Now we can choose the integration contour, and the rules are the same as before: the right poles (from $\Gamma(\cdots - z)$) are to the right of the contour and the left poles (from $\Gamma(\cdots + z)$) are to left.

This representation can be used to evaluate any integral of this family in a Laurent expansion in ϵ.

Case 1.1. In particular, for $I_2^{(1)[d]}(q^2, m^2; 2, 1, d)$, we obtain:

$$I_2^{(1)[d]}(q^2, m^2; 2, 1, d) = \frac{i\pi^{d/2}(-1)^3\,\Gamma(1-\epsilon)}{\Gamma(2)\Gamma(1)(-q^2)^{1+\epsilon}} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \Gamma(1+\epsilon+z) \frac{\Gamma(\epsilon-z)\Gamma(-z)}{\Gamma(1-2\epsilon-z)} \tag{12}$$

and, at $d = 4$ ($\epsilon \to 0$) come to:

$$\begin{aligned} I_2^{(1)[4]}(q^2, m^2; 2, 1, 4) &= \frac{i\pi^2}{q^2} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(1+z)\Gamma(-z)^2}{\Gamma(1-z)} &(13)\\ &= -\frac{i\pi^2}{q^2} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \Gamma(z)\Gamma(-z) &(14)\end{aligned}$$

with an integration contour at $-1 < \mathrm{Re}\{z\} < 0$, and in the last step we used the gamma property to write:

$$\Gamma(1+z) = z\Gamma(z) \qquad , \qquad \Gamma(1-z) = (-z)\Gamma(-z).$$

Here comes the power of this method: if we look at (14) we observe that there is a product $\Gamma(z)\Gamma(-z)$ which would be bad if it was present from the beginning because we could not satisfy our agreement about choosing the integration contours. Indeed, here the right and left poles at $\epsilon = 0$ glue together and there is no space between them, so we can choose a contour that separates the two regions of poles.

However, the situation is unambiguous because we have fixed an integration contour with $-1 < \mathrm{Re}\{z\} < 0$ and we are free to perform identical transformations of the integrand after that. A moral of this discussion is the recipe to derive the MB representation for general powers of the propagators a_l and fix appropriate integration contours at this point. Then, for concrete integer indices a_l, we are allowed to make transformations like:

$$\Gamma(1+z)\Gamma(-z) = \Gamma(z)\Gamma(1-z)$$

but it is necessary to remember about the choice of the contours made before this.

At this point the integral (14) can be evaluated, according to the Cauchy theorem, by closing the integration contour to the right and taking a series of residues (with the minus sign, of course, because we are moving clockwise) at the points $z = 0, 1, 2, \ldots$ (poles of $\Gamma(\cdots - z)$). The residue at $z = 0$ gives:

$$\frac{i\pi^2}{q^2} \ln\left(\frac{-q^2}{m^2}\right) \tag{15}$$

and the residues at $z = 1, 2, \ldots$ give the series:

$$-\frac{i\pi^2}{q^2} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{m^2}{q^2}\right)^n \tag{16}$$

so we get:

$$\begin{aligned}
I_2^{(1)[4]}(q^2, m^2; 2, 1, 4) &= -\frac{i\pi^2}{q^2}\left[-\ln\left(\frac{-q^2}{m^2}\right) - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(-\frac{m^2}{q^2}\right)^n\right] &(17)\\
&= -\frac{i\pi^2}{q^2}\left[-\ln\left(\frac{-q^2}{m^2}\right) - \ln\left(1 - \frac{m^2}{q^2}\right)\right] &(18)\\
&= \frac{i\pi^2}{q^2} \ln\left[\frac{-q^2}{m^2} \cdot \left(1 - \frac{m^2}{q^2}\right)\right] &(19)\\
&= \frac{i\pi^2}{q^2} \ln\left(1 - \frac{q^2}{m^2}\right) &(20)
\end{aligned}$$

which is the well-known result (see Appendix A.2).

Case 1.2. If we take (6), but in the case where the indices are equal to one (so we consider the effective diagram 5) we get (from (11)):

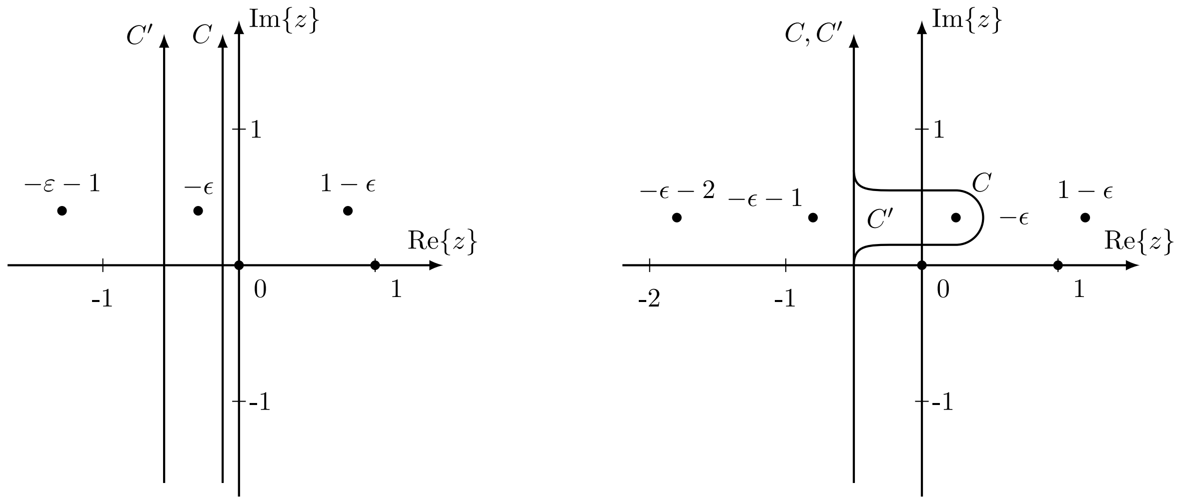
$$\begin{aligned}
I_2^{(1)[4]}(q^2, m^2; 1, 1, d) &= I_2^{(1)[4]}(q^2, m^2; d) &(21)\\
&= \frac{i\pi^{d/2}\,\Gamma(1-\epsilon)}{(-q^2)^\epsilon} \frac{1}{2\pi i} \int_C \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \Gamma(\epsilon + z)\, \frac{\Gamma(1-\epsilon-z)\Gamma(-z)}{\Gamma(2-2\epsilon-z)}. &(22)
\end{aligned}$$

To evaluate MB integrals in a Laurent expansion in ϵ the first point is to analyse how singularities are generated. We know in advance that the given integral has a pole in ϵ because the diagram is UV-divergent. There are no explicit functions with singularities in ϵ so that the pole has to be generated by the MB integration. Indeed, the product $\Gamma(\epsilon + z)\Gamma(-z)$ generates a singularity in ϵ when $\epsilon \to 0$ because the first left pole, i.e. at $z = -\epsilon$, and the first right pole, i.e. $z = 0$, glue together when $\epsilon = 0$, and there is no place for a contour between these poles.

Possible integration contours C in the cases $\mathrm{Re}\{\epsilon\} > 0$ and $\mathrm{Re}\{\epsilon\} < 0$ are shown in figure 2a and 2b, respectively. In the former case, a contour can be chosen as a straight line parallel to the imaginary axis, while in the latter case, there is no such choice. However, no matter which value of ϵ we can imagine, we shall use the same procedure to reveal the pole in ϵ: we write down the integral (22) as the sum of a similar integral over a new contour, C', which goes to the left of the pole at $z = -\epsilon$ and the residue at this point. So, we can spit the integral, via analytic continuation, as:

$$\int_C = \int_{C'} + 2\pi i \,\mathrm{Res}\{\ldots\}_{z=-\epsilon}. \tag{23}$$

In the integral over the shifted contour, the nature of the pole at $z = -\epsilon$ changes, and it becomes right, rather than left, in our terminology.



(a) Possible integration contour in the case $\mathrm{Re}\{\epsilon\} > 0$.

(b) Possible integration contour in the case $\mathrm{Re}\{\epsilon\} < 0$.

Figure 2

The crucial point is that, in the integral over C', we can safely expand the integrand in a Laurent series in ϵ (in this particular example, this is just a Taylor series). As to the residue, it is equal to:

$$i\pi \frac{\Gamma(\epsilon)}{(m^2)^\epsilon (1-\epsilon)} \tag{24}$$

and can explicitly be expanded in ϵ, where, as the convention in Smirnov [2], in order to avoid the Euler's constant γ_E, we will pull out the factor $e^{\gamma_E \epsilon}$ per loop. See the `Mathematica` notebook (residue 1 in *MB representation 1loop computation* section) where we get:

$$\frac{i\pi}{\epsilon} - i\pi\left[\gamma_E - 1 + \ln(m^2)\right] + \mathcal{O}(\epsilon).$$

For the integral over the shifted contour C', with $-1 < \mathrm{Re}\{z\} < 0$, we obtain, at $\epsilon = 0$ (we can directly put $\epsilon = 0$ because the integral converges):

$$i\pi^2 \frac{1}{2\pi i} \int_{C'} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(z)\Gamma(-z)}{1-z} \tag{25}$$

This MB integral can be evaluated by closing the integration contour to the right in the complex z-plane, as in the previous example. See the `Mathematica` notebook (residue 2 in *MB representation 1loop computation* section), where we get the analytic integral result:

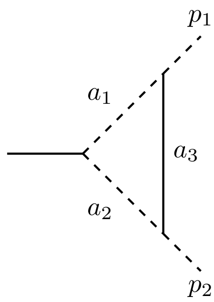
$$\begin{aligned} &= 1 + \left(1 - \frac{m^2}{q^2}\right) \ln\left(\frac{m^2}{-q^2} \frac{1}{1 - \frac{m^2}{q^2}}\right) \\ &= 1 - \left(1 - \frac{m^2}{q^2}\right) \ln\left(1 - \frac{q^2}{m^2}\right). \end{aligned}$$

Combining the corresponding result with the residue calculated above we arrive at the well-known result (see Appendix A.2):

$$I_2^{(1)[d]}(q^2, m^2; d) = i\pi^{d/2} e^{-\gamma\epsilon} \left[\frac{1}{\epsilon} - \ln m^2 + 2 - \left(1 - \frac{m^2}{q^2}\right) \ln\left(1 - \frac{q^2}{m^2}\right) + \mathcal{O}(\epsilon) \right]. \tag{26}$$

In fact, we could similarly proceed by moving the contour C across the right pole at $z = 0$ and, correspondingly, taking minus residue at this point. Then the integral over the new contour C' would be at $0 < \mathrm{Re}\{z\} < 1$.

Example 2. *Triangle diagram.* Now, let's see the case of the diagram:



$$= \int \frac{\mathrm{d}^d k}{(k^2 - 2p_1 \cdot k)^{a_1} \, (k^2 - 2p_2 \cdot k)^{a_2} \, (k^2 - m^2)^{a_3}} \tag{27}$$

where we are taking the masses $(0, 0, m)$, general indeces a_i and the external momentum $p_1^2 = p_2^2 = 0$, considering $q = p_1 - p_2$ so $q^2 = -2p_1 \cdot p_2$, for which we consider:

$$(p_i - k)^2 = k^2 + p_i^2 - 2p_i \cdot k = k^2 - 2p_i \cdot k \qquad , \quad i = 1, 2.$$

We can exploit, again, the MB representation in the simplest way, i.e. apply (4) to the only massive propagator in (27), and evaluate the resulting massless triangle integral:

$$\begin{aligned} I_2^{(1)[d]}(q^2, m^2; & a_1, a_2, a_3, d) = \\ &= \int \frac{\mathrm{d}^d k}{(k^2 - 2p_1 \cdot k)^{a_1} \, (k^2 - 2p_2 \cdot k)^{a_2}} \frac{(-1)^{a_3}}{\Gamma(a_3)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z \, \Gamma(a_3 + z)\Gamma(-z) \frac{(m^2)^z}{(-k^2)^{a_3+z}} && (28) \\ &= \frac{(-1)^{a_3}}{2\pi i \, \Gamma(a_3)} \int_{-i\infty}^{+i\infty} \mathrm{d}z \, (m^2)^z \Gamma(a_3 + z)\Gamma(-z) \int \frac{\mathrm{d}^d k}{(k^2 - 2p_1 \cdot k)^{a_1} (k^2 - 2p_2 \cdot k)^{a_2} (-k^2)^{a_3+z}} && (29) \end{aligned}$$

now we can use the general integral (see the `Mathematica` notabook in the chapter *Table of integral* the integral 1), again with $q = p_1 - p_2$:

$$\begin{aligned} \int \frac{\mathrm{d}^d k}{(-k^2 + 2p_1 \cdot k)^{\lambda_1} (-k^2 + 2p_2 \cdot k)^{\lambda_2} (-k^2)^{\lambda_3}} &= i\pi^{d/2} \frac{\Gamma(-\lambda_1 - \lambda_3 - \epsilon + 2)\Gamma(-\lambda_2 - \lambda_3 - \epsilon + 2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(-\lambda_1 - \lambda_2 - \lambda_3 - 2\epsilon + 4)} \times \\ &\quad \times \frac{\Gamma(\lambda_1 + \lambda_2 + \lambda_3 + \epsilon - 2)}{(-q^2)^{\lambda_1 + \lambda_2 + \lambda_3 + \epsilon - 2}} \end{aligned}$$

collecting the minus sing and using the parameters $\lambda_1 = a_1$, $\lambda_2 = a_2$, $\lambda_3 = a_3$:

$$\begin{aligned} I_2^{(1)[d]}(q^2, m^2; a_1, a_2, a_3, d) &= \frac{(-1)^{a_1+a_2+a_3} i\pi^{d/2}}{\prod_{l=1}^{3} \Gamma(a_i) \, (-q^2)^{a_1+a_2+a_3+\epsilon-2}} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \Gamma(a_3 + z)\Gamma(-z) \times \\ &\times \frac{\Gamma(-a_1 - a_3 - \epsilon + 2 - z)\Gamma(-a_2 - a_3 - \epsilon + 2 - z)\Gamma(a_1 + a_2 + a_3 + \epsilon - 2 + z)}{\Gamma(-a_1 - a_2 - a_3 - 2\epsilon + 4 - z)}. \end{aligned} \tag{30}$$

Case 2.1. We can put all the power of the propagator equal to one:

$$I_2^{(1)[d]}(q^2, m^2; 1, 1, 1, d) = \frac{-i\pi^{d/2}}{(-q^2)^{1+\epsilon}} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \Gamma(1 + z)\Gamma(-z) \frac{\Gamma(-\epsilon - z)^2 \Gamma(1 + \epsilon + z)}{\Gamma(1 - 2\epsilon - z)} \tag{31}$$

which is pretty good because if we want to calculate this integral at $\epsilon = 0$, we observe that we can safely set $\epsilon = 0$ ($d = 4$) in the integrand because the right and left poles in the complex z-plane are well separated. We obtain:

$$I_2^{(1)[d]}(q^2, m^2; 4) = \frac{-i\pi^2}{(-q^2)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \Gamma(-z) \frac{\Gamma(-z)^2\Gamma(1+z)^2}{\Gamma(1-z)} \tag{32}$$

$$= \frac{i\pi^2}{(-q^2)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(-z)^2\Gamma(1+z)^2}{z} \tag{33}$$

where the integration contour can be chosen with $-1 < \mathrm{Re}\{z\} < 0$. The integral can be evaluated by the same procedure as before (you can see the residue theorem in the `Mathematica` notebook, section *triangle* case 1) to find the result (in Appendix A.3 there are the steps to solve it with Feynman parameters):

$$I_2^{(1)[d]}(q^2, m^2; 4) = \frac{i\pi^2}{(-q^2)} \left[-\frac{\pi^2}{3} + \mathrm{Li}_2(x) - \frac{1}{2}\ln^2\left(\frac{m^2}{-q^2}\right) + \ln\left(\frac{m^2}{-q^2}\right)\ln\left(1 - \frac{m^2}{-q^2}\right) \right] \tag{34}$$

where the function $\mathrm{Li}_2(x)$ is the dilogarithm[2].

Case 2.2. We can evaluate the integral (27) with any choise of integer indeces using (30). For example we can choose $(a_1, a_2, a_3) = (2, 1, 1)$ and get:

$$I_2^{(1)[d]}(q^2, m^2; 2, 1, 1, d) = \frac{i\pi^{d/2}}{(-q^2)^{2+\epsilon}} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \Gamma(1+z)\Gamma(-z) \times \\ \times \frac{\Gamma(-1-\epsilon-z)\Gamma(-\epsilon-z)\Gamma(2+\epsilon+z)}{\Gamma(-2\epsilon-z)}. \tag{36}$$

We know in advance that there should be an IR pole in ϵ because of the second power of the first massless propagator, so that we anticipate that a pole is generated by the MB integration. Indeed, we observe that the only source of the singularity in ϵ is the product $\Gamma(1+z)\Gamma(-1-\epsilon-z)$. When $\epsilon \to 0$ the first left pole (from $\Gamma(1+z)$) and the first right pole (from $\Gamma(-1-\epsilon-z)$) tend to each other (glue) and there is no place for an integration contour to go between them.

To evaluate the integral in expansion in ϵ we apply the strategy formulated above (see case 1.2): we turn to the integral over a shifted contour which goes to the left of the first pole of $\Gamma(-1-\epsilon-z)$ so that this pole changes its nature, i.e. becomes left[3]. According to the Cauchy theorem, (36) equals the integral over the shifted contour minus residue of the integrand at the point $z = -1-\epsilon$. Then the integral is evaluated by closing the contour (which can again be taken at $-1 < \mathrm{Re}\{z\} < 0$) to the right and summing up a series of residues at the points $z = 0, 1, 2, \ldots$. We thus obtain (you can see the residue theorem in the `Mathematica` notebook, section *triangle* case 2):

$$I_2^{(1)[d]}(q^2, m^2; 2, 1, 1, d) = -\frac{i\pi^{d/2}e^{-\gamma_E\epsilon}}{(-q^2)} \left[\frac{1}{m^2}\left(\frac{1}{\epsilon} - \ln\left(m^2\right)\right) + \frac{1}{m^2 - q^2}\ln\left(\frac{m^2}{q^2}\right) + \mathcal{O}(\epsilon) \right]. \tag{37}$$

2.2 Hints on evaluating multiple MB integrals

The word *multiple* will mean, in examples below, the number of MB integrations from two to eight (and even ten, in some restricted sense) which is indeed a big number. Still even in such situations, an explicit integration becomes possible, probably, because multiple MB integrals arising in the evaluation of Feynman integrals are very flexible, both in the procedure of resolving the structure of singularities in ϵ and when evaluating finite integrals after expansion in ϵ.

The procedure is the same (generalized) that we have saw for one loop and one MB integral. So, the first step of the method is to derive an appropriate MB representation. Of course, it is advantageous to have a

[2]The polylogarithms function is defined as:

$$\mathrm{Li}_a(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^a} = \frac{(-1)^a}{(a-1)!} \int_0^1 \frac{\ln^{a-1} t}{t - 1/z} \mathrm{d}t. \tag{35}$$

[3]As before, we again had two options: to change the nature of the first pole of $\Gamma(-1-\epsilon-z)$ or the first pole of $\Gamma(1+z)$. Let us agree, for definiteness, that we shall always try to obtain MB integrals expanded in ϵ at $-1 < \mathrm{Re}\{z\} < 0$.

minimal number of MB integrations. In every case, we shall derive MB representations for general powers of the propagators. This is useful and important for several reasons. First, if we obtain a MB representation for general indices, which we might imagine as complex, we will certainly have unambiguous prescriptions for choosing integration contours. Second, such general formulae can be checked using various partial simple cases. Finally, starting from a general formula we can derive a lot of formulae by setting some indices to zero and thereby turning to graphs where the corresponding lines are contracted to a point.

As we know, in the second step, one resolves the singularity structure in ϵ, taking residues and shifting contours, with the goal to obtain a sum of integrals where one can expand integrands in Laurent series in ϵ.

In fact, up to now, we were dealing with one-parametric MB integrals. To resolve the singularities in ϵ we analysed the integrand, and then shifted contours and took residues, in an appropriate way. In the end of this procedure we obtained either explicit expressions for general ϵ or integrals where a Laurent expansion of the integrand in ϵ was possible. In fact, the strategy for a multiple MB integration is a generalization of this procedure, which arise when evaluating more complicated Feynman integrals. Of course, the resolution of singularities in ϵ in such multi-dimensional MB integrals is more complicated than in the one-dimensional case. Usually, the poles in ϵ are *not visible at once*, at a first integration over one of the MB variables. However, the rule for finding a mechanism of the generation of poles is just a straightforward generalization of the rule used in the previous one-loop examples with one-parametric MB integrals. For example, we can observe that the product of $\Gamma(1+z)$ and $\Gamma(-1-\epsilon-z)$ generated a pole of the type $\Gamma(-\epsilon)$ (after the integration over z), and this is nothing but the value of one of these gamma functions at the pole of the other gamma function (this happens because the poles of the two gammas hurt each other and capture the contour).

Suppose that we are dealing with a multiple MB integral and we start from the integration over one of the variables, z. We shall analyse various products like $\Gamma(a+z)\Gamma(b-z)$, where a and b *depend* on the rest of the variables, with the understanding that this integration generates a pole of the type $\Gamma(a+b)$.

Indeed, if we shift an initial contour of integration over z across the point $z=-a$ we obtain two factors: an integral over a new contour which is not singular at $a+b=0$, and a corresponding residue (of the pole) involves an explicit factor $\Gamma(a+b)$. Well, sometimes it turns out that it is cancelled by a factor in the denominator.

This observation shows that any contour of one of the next integrations over the rest of the MB variables should be chosen according to this dependence, $\Gamma(a+b)$. We continue this analysis, in a similar way, with various next integrations of the second level, etc. In other words, we consider various orders of integrations over given MB variables and analyse whether a singular dependence on ϵ in the form of some gamma function, e.g. $\Gamma(-\epsilon)$, is generated in a given order.

After this first step, we can identify some gamma functions (in the numerator of the integrand) that are essential for the generation of poles in ϵ. Then we proceed with one of the MB integrations as in the case of one-dimensional MB integrals by shifting contours and taking residues. In the integral over the shifted contour, we continue this procedure by taking care of another key gamma function etc. The corresponding residue has one integration less. We deal with it exactly like with the initial integral, i.e. perform an analysis of generation of poles and then shift contours and take residues. In the end of our procedure, we are left with MB integrals which can be expanded in a Laurent series in ϵ under the sign of integration.

The third step of the method is to evaluate integrals expanded in ϵ after the second step. Here one can use corollaries of the first and the second Barnes lemmas:

$$\frac{1}{2\pi i}\int_{-i\infty}^{+\infty} \mathrm{d}z\, \Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3-z)\Gamma(\lambda_4-z) = \\ = \frac{\Gamma(\lambda_1+\lambda_3)\Gamma(\lambda_1+\lambda_4)\Gamma(\lambda_2+\lambda_3)\Gamma(\lambda_2+\lambda_4)}{\Gamma(\lambda_1+\lambda_2+\lambda_3+\lambda_4)} \tag{38}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{+\infty} \mathrm{d}z\, \frac{\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3-z)\Gamma(\lambda_4-z)\Gamma(\lambda_5-z)}{\Gamma(\lambda_6+z)} = \\ = \frac{\Gamma(\lambda_1+\lambda_4)\Gamma(\lambda_2+\lambda_4)\Gamma(\lambda_3+\lambda_4)\Gamma(\lambda_1+\lambda_5)\Gamma(\lambda_2+\lambda_5)\Gamma(\lambda_3+\lambda_5)}{\Gamma(\lambda_1+\lambda_2+\lambda_4+\lambda_5)\Gamma(\lambda_1+\lambda_3+\lambda_4+\lambda_5)\Gamma(\lambda_2+\lambda_3+\lambda_4+\lambda_5)} \tag{39}$$

respectively (a table of these formulae is presented in Appendix D of Smirnov [2]) and with $\lambda_6 = \sum_{i=1}^{5} \lambda_i$. Typically, the integration over the last variable is performed, as in the previous examples, by shifting the contour to the right (or left) and taking a series of residues (there is the Appendix C of Smirnov [2] where you can find some series summed).

In fact, we are going to be pragmatic and not bother whether the change of the order of integration over MB variables is legitimate.[4] Usually, at least at large values in the complex plane, the convergence of MB integrals is perfect because gamma functions have exponential decrease in both imaginary directions. This property can be used for numerical checks. Moreover, in complicated situations, one can decompose a given integrand into pieces and choose an order of integration for every piece in a special way, with the possibility to integrate explicitly (table formulae of Appendix D in Smirnov [2]).

We shall apply some standard properties of integration for multiple MB integrals. We shall use changes of variables of the type $z \to \pm z + z_0$. When doing this we shall, of course, trace how the nature of various poles is transformed. Note that, after such a change, $z \to -z$, right poles become left poles.

Note. The IBP is also possible in multiple MB integrals, although it is reasonable to apply it in rare situations. Still sometimes it is useful. For example, tabulated formulae of Appendix D with the factor $1/z^2$ were derived using the IBP identity:

$$\int_C \mathrm{d}z \, \frac{f(z)}{z^2} = \int_C \mathrm{d}z \, \frac{f'(z)}{z} \tag{40}$$

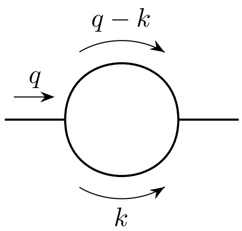
where we are able to decrease the power of the denominator, but with the cost to have a derivative in the numerator.

An althernative strategy. First, one chooses a domain of the regularization parameter ϵ and values of the real parts of the integration variables, $z_i, w, \dots$ in such a way that all the integrations over the MB variables can be performed over straight lines parallel to imaginary axis. In fact this is not always possible. However, in such a situation, one can introduce auxiliary analytic regularization to provide the existence of such straight contours. Then one tends ϵ to zero, and whenever a pole of some gamma function is crossed one takes into account the corresponding residue. (If the auxiliary analytic regularization was introduced, one first performs, in a similar way, the analytic continuation to zero values of the corresponding analytic parameters.) It is simple to organize this procedure in such a way that no more than one pole is crossed at the same time. For every resulting residue, which involves one integration less, a similar procedure is applied, and so on.

You can see the end of the chapter 4 of Smirnov [2] where talk about the discussion of the Czakon's code. Indeed, this strategy is simpler to be implemented via numerical code.

2.3 One-loop example with multiple MB integral

See exmple 3.5 (and extension in 4.3) in Smirnov [2]. We want to consider the following Feynman diagram/integral:



$$= \int \frac{\mathrm{d}^d k}{(k^2 - m^2)[(q-k)^2 - m^2]} \equiv J \tag{41}$$

where we can use, two times, the relation (4) to get:

$$\begin{aligned} J = &\int \mathrm{d}^d k \, \frac{1}{2\pi i} \int_{-i\infty}^{+\infty} \mathrm{d}z_1 \, \Gamma(1+z_1)\Gamma(-z_1) \frac{(m^2)^{z_1}}{(k^2)^{1+z_1}} \times \\ &\times \frac{1}{2\pi i} \int_{-i\infty}^{+\infty} \mathrm{d}z_2 \, \Gamma(1+z_2)\Gamma(-z_2) \frac{(m^2)^{z_2}}{[(q-k)^2]^{1+z_2}} \end{aligned} \tag{42}$$

$$\begin{aligned} = &\left(\frac{1}{2\pi i}\right)^2 \int_{-i\infty}^{+i\infty} \mathrm{d}z_1 \int_{-i\infty}^{+i\infty} \mathrm{d}z_2 \, \Gamma(1+z_1)\Gamma(-z_1)\Gamma(1+z_2)\Gamma(-z_2) \, (m^2)^{z_1+z_2} \times \\ &\times \int \frac{\mathrm{d}^d k}{(k^2)^{1+z_1}[(q-k)^2]^{1+z_2}} \end{aligned} \tag{43}$$

[4]In fact, the crucial point is not the convergence of the integral in the basic identity (3), but the interchange of the order of integrations between the Mellin-Barnes integral and the parameter integrals.

but the integral in the last row is well known from (9), where we collect minus sign and write $\lambda_1 = 1 + z_1$ and $\lambda_2 = 1 + z_2$, so we obtain:

$$
\begin{aligned}
J &= \left(\frac{1}{2\pi i}\right)^2 \int_{-i\infty}^{+i\infty} \mathrm{d}z_1 \int_{-i\infty}^{+i\infty} \mathrm{d}z_2 \, \Gamma(1+z_1)\Gamma(-z_1)\Gamma(1+z_2)\Gamma(-z_2)\,(m^2)^{z_1+z_2} \times \\
&\quad \times (-1)^{2+z_1+z_2} i\pi^{d/2} \frac{\Gamma(z_1+z_2+\epsilon)\Gamma(1-z_1-\epsilon)\Gamma(1-z_2-\epsilon)}{\Gamma(1+z_1)\Gamma(1+z_2)\Gamma(2-z_1-z_2-2\epsilon)} \frac{1}{(-q^2)^{z_1+z_2+\epsilon}} \qquad (44) \\
&= \frac{i\pi^{d/2}}{(-q^2)^\epsilon (2\pi i)^2} \int_{-i\infty}^{+i\infty} \mathrm{d}z_1 \int_{-i\infty}^{+i\infty} \mathrm{d}z_2 \left(\frac{m^2}{q^2}\right)^{z_1+z_2} \times \\
&\quad \times \frac{\Gamma(-z_1)\Gamma(-z_2)\Gamma(z_1+z_2+\epsilon)\Gamma(1-z_1-\epsilon)\Gamma(1-z_2-\epsilon)}{\Gamma(2-z_1-z_2-2\epsilon)}. \qquad (45)
\end{aligned}
$$

We can perform the integral via `Mathematica` (see the section *Multi-MB integral* in the attached notebook) to get the result:

$$
J = i\pi^{d/2}\Gamma(\epsilon)\, e^{-\gamma_E \epsilon} \frac{2^{2\epsilon}}{(-q^2)^\epsilon} \left[\frac{1}{\epsilon} + 2 - \ln\left(4\frac{m^2}{-q^2}\right) - \sqrt{1 - 4\frac{m^2}{q^2}} \ln\left(\frac{\sqrt{1-4\frac{m^2}{q^2}}+1}{\sqrt{1-4\frac{m^2}{q^2}}-1} \right) \right]. \qquad (46)
$$

2.4 Conclusion

To summarize, the method of Mellin-Barnes (MB) representation is a powerful tool which has good chances to be developed and optimized further. It can even happen that it will provide the first possibility to calculate any individual Feynman integral, at the high level of complexity of modern calculations, with a reasonable precision. This will be a very important option in situations where the reduction to master integrals (see the next section) turns out to be too complicated.

On a more practical level, as demonstrated through the examples analyzed in this work, the MB approach proves to be highly effective even when dealing with simpler configurations. By focusing on one-loop diagrams resulting in just one or two MB integrations, we can clearly appreciate the core mechanics of the method without the burden of overwhelming technical complexities. These foundational cases not only illustrate how the non-trivial radial and angular dependencies of Feynman integrals can be elegantly decoupled, but they also serve as a crucial stepping stone. Mastering these low-dimensional representations is essential for building the intuition and analytical framework required to tackle the multi-loop, multi-scale challenges of modern perturbative quantum field theory.

3 IBP and Reduction to master integrals

In this section we want to introduce another method to solve Feynman integrals, which is based on integration by parts (IBP) within dimensional regularization. We know that in this regularization scheme we have the property:

$$\int \mathrm{d}^d k_1 \cdots \int \mathrm{d}^d k_h \, \frac{\partial}{\partial k_i^\mu} \prod_{l=1}^{L} \tilde{D}_{F,l}(p_l) = 0 \qquad , \quad i = 1, \ldots, h \tag{47}$$

where $\tilde{D}_{F,i}(p_i)$ is the Fourier tranformation of propagator (in a generic form, so with $Z_i(p_i)$ a polynomial and $a_i = \{1, 2\}$):

$$\tilde{D}_{F,i}(p_i) = \int \mathrm{d}^4 x \, e^{ip_i \cdot x} D_{F,i}(x) = \frac{iZ_i(p_i)}{(p_i^2 - m_i^2 + i\emptyset)^{a_1}}.$$

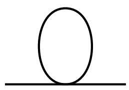
The idea is to write down various equations (47) for integrals of derivatives with respect to loop momenta and use this set of relations between Feynman integrals in order to solve the reduction problem, i.e. to find out how a general Feynman integral of the given class can be expressed linearly in terms of some *master integrals* (so some fundamental integrals).

In contrast to the evaluation of the master integrals, which is performed, at a sufficiently high level of complexity, in a Laurent expansion in ϵ, the reduction problem is usually solved at general d, and the expansion in ϵ does not provide simplifications here.

The reduction can be stopped whenever one arrives at sufficiently simple integrals. On the other hand, one could try to solve the reduction problem in the ultimate mathematical sense, i.e. to reduce a given integral to true irreducible integrals which cannot be reduced further.

Our goal is to show how the IBP works, through a one-loop example, just to focus on the Baikov's reduction method in the next section (see §3.1).

Example 3. Let's focus on the tadpole:



$$= \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^a} \equiv J_a \tag{48}$$

and use the identity (47):

$$\int \mathrm{d}^d k \, \frac{\partial}{\partial k} \cdot k \frac{1}{(k^2 - m^2)^a} = 0 \tag{49}$$

where we write $(\partial/\partial k) \cdot k = (\partial/\partial k^\mu) k^\mu$. Now we wanto to write the resulting quantities in terms of integrals (48). Let's start with:

$$\frac{\partial}{\partial k} \cdot k \frac{1}{(k^2 - m^2)^a} = \frac{\partial k^\mu}{\partial k^\mu} \cdot \frac{1}{(k^2 - m^2)^a} + k^\mu \frac{\partial}{\partial k^\mu} \frac{1}{(k^2 - m^2)^a} \tag{50}$$

$$= \delta^\mu_\mu \frac{1}{(k^2 - m^2)^a} + k^\mu (-a) \frac{1}{(k^2 - m^2)^{a+1}} \frac{\partial k^2}{\partial k^\mu} \tag{51}$$

$$= d \frac{1}{(k^2 - m^2)^a} - a \frac{k^\mu}{(k^2 - m^2)^{a+1}} \frac{\partial k^\nu k_\nu}{\partial k^\mu} \tag{52}$$

$$= d \frac{1}{(k^2 - m^2)^a} - 2a \frac{k^2}{(k^2 - m^2)^{a+1}} \tag{53}$$

so we get:

$$\int \mathrm{d}^d k \left[d \frac{1}{(k^2 - m^2)^a} - 2a \frac{k^2}{(k^2 - m^2)^{a+1}} \right] = 0 \tag{54}$$

$$dJ_a - 2a \int \mathrm{d}^d k \, \frac{k^2}{(k^2 - m^2)^{a+1}} = 0. \tag{55}$$

Now, we can replace k^2 with:

$$k^2 \longrightarrow (k^2 - m^2) + m^2 \tag{56}$$

to obtain:

$$dJ_a - 2a \int \mathrm{d}^d k \, \frac{(k^2 - m^2) + m^2}{(k^2 - m^2)^{a+1}} = 0 \tag{57}$$

$$(d - 2a) J_a - 2am^2 J_{a+1} = 0 \tag{58}$$

that gives us the recurrence relation:

$$J_a = \frac{d - 2a + 2}{2(a-1)m^2} J_{a-1}. \tag{59}$$

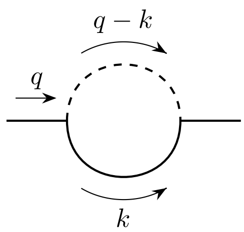
So, we see that any Feynman integral with integer $a > 1$ can be expressed recursively in terms of one integral J_1 which we therefore consider as a *master integral*. (Observe that all the integrals with non-positive integer indices are zero since they are massless tadpoles.) This can be done explicitly here[5]:

$$J_a = \frac{(-1)^a (1 - \frac{d}{2})_{a-1}}{(a-1)!(m^2)^{a-1}} J_1 \tag{61}$$

with the master integral (which we derived during the MSc course):

$$J_1 = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)} = -i\pi^{d/2} \Gamma \left(1 - \frac{d}{2}\right) (m^2)^{d/2-1}. \tag{62}$$

Example 4. Now consider:



$$= \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^{a_1} [(q-k)^2]^{a_2}} \equiv F(a_1, a_2) \tag{63}$$

where we can notice that if $a_1 \leq 0$ the integral is zero, because it's a massless tadpole which is naturally put to zero within dimensional regularization (if $a_1 \leq 0$ the massive propagator is at the numerator or is 1). Moreover, any integral with $a_2 \leq 0$ can be evaluated in terms of gamma functions for general d because the massless propagator dissapear and we have just a massive tadpole (which is well known). We need to put:

$$\frac{\partial}{\partial k^\mu} k^\mu f(a_1, a_2) = 0 \quad , \quad \frac{\partial}{\partial k^\mu} q^\mu f(a_1, a_2) = 0 \tag{64}$$

where $f(a_1, a_2)$ is the integrand in (63). We can easily see that:

$$\frac{\partial}{\partial k^\mu} f(a_1, a_2) = \frac{\partial}{\partial k^\mu} \left(\frac{1}{(k^2 - m^2)^{a_1} [(q-k)^2]^{a_2}} \right) \tag{65}$$

$$= -2a_1 k_\mu f(a_1 + 1, a_2) - 2a_2 (q_\mu - k_\mu) f(a_1, a_2 + 1) \tag{66}$$

so the relations (64) became:

$$\frac{\partial}{\partial k^\mu} \left(k^\mu f(a_1, a_2) \right) = df(a_1, a_2) + k^\mu \frac{\partial f(a_1, a_2)}{\partial k^\mu} \tag{67}$$

$$\frac{\partial}{\partial k^\mu} \left(q^\mu f(a_1, a_2) \right) = q^\mu \frac{\partial f(a_1, a_2)}{\partial k^\mu} \tag{68}$$

[5]Where we write $(x)_a$ the Pochhammer symbol, defined as:

$$(x)_a = x(x+1)(x+2) \ldots (x+a-1). \tag{60}$$

which easly bring us to[6]:

$$d - 2a_1 - a_2 - 2m^2 a_1 \mathbf{1}^+ - a_2 \mathbf{2}^+ (\mathbf{1}^- - q^2 + m^2) = 0 \tag{69}$$
$$a_2 - a_1 - a_1 \mathbf{1}^+ (q^2 + m^2 - \mathbf{2}^-) - a_2 \mathbf{2}^+ (\mathbf{1}^- - q^2 + m^2) = 0 \tag{70}$$

where the standard notation for increasing and lowering operator has been used:

$$\mathbf{1}^+ \mathbf{2}^- F(a_1, a_2) = F(a_1 + 1, a_2 - 1).$$

We can do the difference between (69) multiplied by $(q^2 + m^2)$ and (70) multiplied by $2m^2$:

$$(q^2 - m^2)^2 a_2 \mathbf{2}^+ = (q^2 - m^2) a_2 \mathbf{1}^- \mathbf{2}^+ - (d - 2a_1 - a_2) q^2 - (d - 3a_2) m^2 + 2m^2 a_1 \mathbf{1}^+ \mathbf{2}^-. \tag{71}$$

When the left-hand side of (71) is applied to $F(a_1, a_2)$, we obtain integrals with reduced a_2 or, due to the first term on the right-hand side, reduced a_1. So we have a reduction algorithm.

Case 4.1. Indeed, if we suppose that $a_2 = 1$ and we consider the difference of (69) and (70):

$$d - a_1 - 2a_2 - a_1 \mathbf{1}^+ (\mathbf{2}^- - q^2 + m^2) = 0 \tag{72}$$
$$\implies \quad (q^2 - m^2) a_1 \mathbf{1}^+ = a_1 + 2 - d + a_1 \mathbf{1}^+ \mathbf{2}^- \tag{73}$$

we can use this relation to to reduce the index a_1 to one or the index a_2 to zero. We see that we can now express any integral of the given family as a linear combination of the integral $F(1, 1)$ and simple integrals with $a_2 \leq 0$ which can be evaluated for general d in terms of gamma functions. In particular, we have:

$$F(2, 1) = \frac{1}{m^2 - q^2} \Big[(1 - 2\epsilon) F(1, 1) - F(2, 0) \Big]. \tag{74}$$

Case 4.2. Now work with general indeces, but first try to understand how the reduction works. Let us consider what happens when we apply the recurrence relation (71) to an arbitrary pair of positive indices (a_1, a_2):

- Each time the relation is applied, the index a_2 is systematically lowered. By iterating this process, the algorithm will eventually terminate when the second index either reaches the boundary value $a_2 = 1$ or drops to non-positive values $(a_2 \leq 0)$.
- As previously noted, these integrals do not require further reduction because they can be evaluated analytically in terms of Euler Gamma functions for generic d. They are collected into the boundary remainder term, which corresponds to the summation:

  $$\sum c'(i_1, i_2) \, F(i_1, i_2) \tag{75}$$

  with $i_2 \leq 0$.
- During the recursive steps, the shifting term $F(a_1 - 1, \dots)$ might drive the first index to zero or negative values. However, in this kinematic configuration, the integral reduces to a massless tadpole, which vanishes identically within dimensional regularization $(F(a_1 \leq 0, a_2) = 0)$ and drops out of the calculation.
- If we continue to reduce the power of the propagators as much as possible—while avoiding the vanishing massless tadpoles or the analytically solvable massive ones—the only point in the positive quadrant of the index space (a_1, a_2) that *cannot be further reduced* without hitting the aforementioned boundary conditions is the minimal configuration:

$$a_1 = 1 \quad , \quad a_2 = 1. \tag{76}$$

[6] To get these relations we neet to contract the Lorentz indeces and use the substitution:

$$\begin{aligned} &k^2 \longrightarrow (k^2 - m^2) + m^2 \\ &2q \cdot k - 2k^2 = (k^2 - m^2) - (q - k)^2 - m^2 - q^2 \\ &- 2q \cdot k = (q - k)^2 - k^2 - q^2 \\ &2q^2 - 2q \cdot k = -(k^2 - m^2) + (q - k)^2 - m^2 + q^2. \end{aligned}$$

Consequently, any generic initial integral $F(a_1, a_2)$ is systematically *deconstructed* by the recurrence equation (71) until it is expressed as a linear combination of:

- A rational kinematic coefficient $c_1(a_1, a_2)$ multiplying the sole surviving fundamental integral $F(1, 1)$, which is defined as the **Master Integral** (I_1).
- A finite sum of simplified integrals where the second propagator has vanished ($i_2 \leq 0$), which can be computed exactly.

So we can write, as a result of the reduction:

$$F(a_1, a_2) = c_1(a_1, a_2)I_1 + \sum_{\substack{i_1>0 \\ i_2\leq 0}} c'(i_1, i_2)F(i_1, i_2) \tag{77}$$

where the sum is finite and $c'(i_1, i_2)$ are rational functions of q^2, m^2 and d.

While equation (77) provides a pragmatic stopping point, it is incomplete from a strict mathematical perspective, as the boundary framework still contains a non-fixed summation over integrals with $i_2 \leq 0$. To achieve a maximal reduction to truly irreducible structures, one can exploit the IBP relations once more within this substructure. In this way we obtain a real expression with only irreducible integrals, i.e. that cannot be expressed linearly in terms of other integrals.

By analyzing the case where $a_2 \leq 0$, relation (69) can be systematically used to reduce the first index until $a_1 = 1$. At this stage, so with $a_1 = 1$, combining the primary IBP identities (69) multiplied by $\mathbf{2}^-$ to express the term $2m^2 a_1 \mathbf{1}^+\mathbf{2}^-$ in (71), yields the recurrence relation:

$$(d - a_2 - 1)\mathbf{2}^- = (q^2 - m^2)^2 a_2 \mathbf{2}^+ + (q^2 + m^2)(d - 2a_2 - 1). \tag{78}$$

This relation acts as an upgrading operator for the second index, raising a_2 from negative values up to the threshold $a_2 = 0$.

We come to the conclusion that there are two irreducible integrals $I_1 = F(1, 1)$ given by (26) and $I_2 = F(1, 0)$ which equals the right-hand side of the tadpole integral (62), and any integral from our family can be expressed linearly in terms of them. This reduction procedure to I_1 and I_2 can easily be implemented on a computer.

Observe that the integrals I_1 and I_2 cannot be linearly expressed through each other, with a coefficient which is a rational function of d, because, at general d, I_1 is a non-trivial function of q_2 and m_2, while I_2 is independent of q_2. Explicitly, instead of relation (77), we now have:

$$F(a_1, a_2) = c_1(a_1, a_2)I_1 + c_2(a_1, a_2)I_2. \tag{79}$$

We can see that the coefficient $c_2(a_1, a_2)$ could be extracted analytically by computing the Gamma functions via computer algebra systems like `Mathematica`. Indeed, starting again from (77) we can suppose that we made the observation that all the integrals with nonpositive a_2 are proportional to $F(1, 0)$ with coefficients that are rational functions of d. Then we can write down equation (79) immediately and say that the coefficient function at I_2 is obtained as:

$$c_2(a_1, a_2) = \frac{1}{F(1, 0)} \sum_{\substack{i_1>0 \\ i_2\leq 0}} c'(i_2, i_2)F(i_1, i_2) \tag{80}$$

where the integrals on the right-hand side are evaluated by well-know relation (see Appendix A of Smirnov [2]). This ratio can be simplified using well-known properties of gamma functions or, at least for smaller values of the indices, we can proceed with `Mathematica`.

The IBP framework, so using the relation (79), achieves this exact maximal reduction through purely algebraic maneuvers. This bypasses the need for explicit analytical integration, cementing IBP identities as the backbone of automated multi-loop calculations in modern particle physics.

3.1 Reduction to master integrals Baikov's method

To conclude, the Integration by Parts (IBP) method stands as a cornerstone of modern perturbative quantum field theory. As demonstrated through the previous examples, its primary strength lies in its purely algebraic nature: by transforming the problem of continuous integration into a system of discrete linear recurrence relations, it allows for the systematic reduction of vast families of Feynman integrals to a minimal basis of true

Master Integrals, such as $I_1 = F(1,1)$ and $I_2 = F(1,0)$. This algebraic framework forms the backbone of highly successful automated computer algorithms, most notably *Laporta's algorithm*.

However, the standard IBP approach encounters severe limitations when pushed to the high-precision frontiers of modern physics. As the number of loops, legs, or internal mass scales increases, the number of coupled equations grows exponentially and the reduction process becomes practically unfeasible. Furthermore, the traditional IBP relations do not inherently reveal the underlying geometric or topological properties of the Master Integrals, treating them merely as abstract labels in a matrix.

To overcome these computational barriers, alternative reduction strategies have been developed to bypass the generation of large intermediate systems of equations. Chief among these is *Baikov's method of reduction*. Instead of working in the standard momentum space or Feynman parameter space, Baikov's approach introduces a radical change of variables, projecting the loop momenta directly onto the space of the independent scalar products and propagators.

In Baikov's representation, the Feynman integral is rewritten in terms of a specific algebraic polynomial — known as the Baikov polynomial — raised to a d-dependent power. This formulation offers a profound advantage: the IBP identities can be formulated as differential operators acting directly on this polynomial. Consequently, Baikov's method provides a much more direct, localized framework to determine the number of master integrals and solve the reduction problem, paving the way for cutting-edge multi-loop calculations where traditional IBP methods falter.

3.1.1 The General Parametric Representation

Consider a family of h-loop Feynman integrals in d dimensions with N denominators. We can denote this family by:

$$F(\underline{a}) = \int \cdots \int \frac{\mathrm{d}^d k_1 \ldots \mathrm{d}^d k_h}{E_1^{a_1} \ldots E_N^{a_N}} \tag{81}$$

where $\underline{a} = (a_1, \ldots, a_N)$ is a collection of integer indices. The denominators E_r are quadratic or linear functions with respect to the loop momenta k_i:

$$E_r = \sum_{i \geq j \geq 1}^{h} A_r^{ij} k_i \cdot k_j + \sum_{i=1}^{h} B_r^i \cdot k_i + D_r. \tag{82}$$

Here, B_r^i are linear combinations built from independent external momenta $q_1, \ldots, q_n$. The total number of independent scalar products involving loop momenta is $N = h(h+1)/2 + hn$.

The goal of the reduction problem is to express any arbitrary integral $F(\underline{a})$ as a linear combination of a minimal set of true master integrals I_i:

$$F(\underline{a}) = \sum_i c_i(\underline{a}) I_i \tag{83}$$

where the coefficient functions $c_i(\underline{a})$ are rational functions of the spacetime dimension d, masses, and external kinematic invariants. We have the natural normalization conditions:

$$c_i(I_j) = \delta_{ij} \tag{84}$$

which simply mean that any master integral cannot be expressed in terms of other master integrals.

Criteria for Choosing Master Integrals From a purely mathematical standpoint, once the reduction problem is solved, any linearly independent set of integrals can span the vector space of a given topology. For instance, if a system is spanned by $I_1 = F(1,1)$ and $I_2 = F(1,0)$, one could mathematically change the basis to $I_1 = F(1,1)$ and $I_2 = F(2,1)$. See the example 4.

However, from a practical and computational perspective, this is highly unnatural. In the standard workflow of IBP reductions, we follow a strict hierarchical structure where complicated integrals are systematically reduced to simpler ones. To preserve this hierarchy, we adopt the following standard convention: *the master integrals must be chosen such that they contain as many non-positive indices ($a_i \leq 0$) as possible.*

When an index a_i is zero or negative, it means a propagator is either absent or appears as an irreducible numerator. This effectively drops the integral into a simpler sub-sector (or sub-topology). Therefore, when a Feynman integral is referred to as *irreducible*, it is implicitly understood that it cannot be reduced further to simpler integrals possessing a higher number of non-positive indices.

3.1.2 The Vacuum Case and IBP Identities

To understand the core of Baikov's method, let us first restrict our attention to the simpler case of **vacuum Feynman integrals** (where there are no external momenta, meaning $B_r^i = 0$ and $D_r = -m_r^2$). The denominators simplify to:

$$E_r = \sum_{h \geq i \geq j \geq 1} A_r^{ij} k_i \cdot k_j - m_r^2 \tag{85}$$

where $r = 1, \ldots, N = h(h+1)/2$. The traditional IBP equations in the vacuum case originate from the vanishing of total derivatives under dimensional regularization:

$$\int \cdots \int \mathrm{d}^d k_1 \ldots \mathrm{d}^d k_h \frac{\partial}{\partial k_i} \cdot \left(\frac{k_j}{E_1^{a_1} \ldots E_N^{a_N}} \right) = 0, \quad \text{for } i \geq j \tag{86}$$

By performing the differentiation explicitly and re-expressing the resulting scalar products $k_i \cdot k_j$ back in terms of the denominators E_r, as we did in the previous example, we can rewrite the IBP relations in an operational form using index-shifting operators:

$$\sum_{r,r',i'} \bar{A}_r^{i'i} \tilde{A}_{r'}^{ji'} \left(\mathbf{r}'^- + m_{r'}^2 \right) a_r \mathbf{r}^+ = \frac{d-h-1}{2} \delta_{ij} \tag{87}$$

where the shifting operators $\mathbf{r}^+$ and $\mathbf{r}^-$ act on the indices via:

$$\mathbf{r}^+ F(\ldots, a_r, \ldots) = F(\ldots, a_r + 1, \ldots) \tag{88}$$

$$\mathbf{r}^- F(\ldots, a_r, \ldots) = F(\ldots, a_r - 1, \ldots) \tag{89}$$

for the matrix $\bar{A}_r^{ij}$ is valid:

$$\bar{A}_r^{ij} = A_r^{ij} \quad \text{for } i = j \tag{90}$$

$$\bar{A}_r^{ij} = A_r^{ij}/2 \quad \text{for } i > j \tag{91}$$

$$\bar{A}_r^{ij} = A_r^{ji}/2 \quad \text{for } i < j \tag{92}$$

and it's defined as follow. Take the quadratic $N \times N$ matrix A, where the first index is labelled by pairs (i, j) with $i \geq j$, and the second index is r. The corresponding inverse matrix $(A^{-1})_r^{ij}$ (with $i \geq j$) satisfies:

$$\sum_{r=1}^{N} A_r^{ij} (A^{-1})_r^{i'j'} = \delta_{ii'} \delta_{jj'}. \tag{93}$$

Then $\tilde{A}_r^{ij}$ is the symmetrical extension of $(A^{-1})_r^{ij}$ to all values of i, j.

3.1.3 The Baikov Representation and the Basic Polynomial

The essence of Baikov's method is to change variables from the loop momenta k_i directly to the parameters $x_r \equiv E_r + m_r^2$. This transformation yields a remarkably elegant parametric representation for the Feynman integral:

$$F(\underline{a}) = \int \cdots \int \frac{\mathrm{d}x_1 \ldots \mathrm{d}x_N}{x_1^{a_1} \ldots x_N^{a_N}} \left[P(\underline{x}') \right]^{(d-h-1)/2} \tag{94}$$

where the parameters $\underline{x}' = (x_1', \ldots, x_N')$ are obtained from $\underline{x} = (x_1, \ldots, x_N)$ by the shift $x_r' = x_r + m_r^2$. The function $P(\underline{x})$ is known as the **Baikov Polynomial** and it is defined as the determinant of the field-dependent matrix constructed from the system:

$$P(\underline{x}) = \det_{ij} \left(\sum_{r=1}^{N} \tilde{A}_r^{ij} x_r \right) \tag{95}$$

Following there are simple practical prescriptions for evaluating the basic polynomials $P(\underline{x})$:

1. Set up and solve the linear system:

$$\sum_{i \geq j \geq 1} A_r^{ij} k_i \cdot k_j = E_r \quad \text{for } r = 1, \ldots, N \tag{96}$$

with respect to the scalar products $k_i \cdot k_j$ $(i \geq j)$.

2. Replace each instance of E_r on the right-hand side of your solution with the corresponding algebraic variable x_r.

3. Symmetricize the resulting expression to extend the relations to all values of i and j.

4. Construct the matrix from these entries and compute its determinant to yield $P(\underline{x})$.

3.1.4 Generalization to Non-Vacuum Integrals

When we transition from vacuum diagrams to general Feynman integrals involving n independent external momenta $q_1, \ldots, q_n$, the presence of external invariants complicates the direct application of Baikov's change of variables. To handle these non-vacuum configurations, Baikov's method maps the problem back onto a pseudo-vacuum framework.

The Taylor Expansion and Cauchy Residue Approach For a family of propagator-type integrals with a single external momentum q, one can explore the coefficients of a Taylor expansion with respect to q^2:

$$F(q^2; a_1, \ldots, a_N) \sim \sum_{a_{N+1}=1}^{\infty} (q^2 - m_{N+1}^2)^{a_{N+1}-1} F(a_1, \ldots, a_N, a_{N+1}) \tag{97}$$

The auxiliary objects $F(a_1, \ldots, a_N, a_{N+1})$ turn out to satisfy standard vacuum IBP relations. Mathematically, isolating a specific sector or setting specific indices to zero (which corresponds to analyzing irreducible numerators) can be beautifully framed in the complex plane. By treating the integration over the Baikov variables x_j as a Cauchy contour integral around the origin, we can write:

$$\frac{1}{2\pi i} \oint \frac{\mathrm{d}x_j}{x_j^{a_j}} \int \ldots [P(\underline{x})]^{(d-h-1)/2} \tag{98}$$

By Cauchy's residue theorem, if we are only interested in the leading term of the expansion (the threshold where $q^2 = m_{N+1}^2$), the higher-order terms vanish and the extra loop-like parameter x_{N+1} is effectively evaluated at zero. Indeed, according to the Cauchy theorem, this expression reduces to the Taylor expansion of order $a_j - 1$ of the integrand in x_j so that it becomes a linear combination of terms:

$$\int \cdots \int [P_i(\underline{x})]^{z-n_d} \prod_{j: a_{ij} \leq 0} \frac{\mathrm{d}x_j}{x_j^{n_j}} \tag{99}$$

where $z = (d-h-1)/2$ and $P_i(\underline{x})$ is obtained from $P(\underline{x})$ by setting all the variables x_j with j such that $a_{ij} = 1$. We shall use n_j instead of a_j for powers of x_j in auxiliary parametric integrals. Observe that the parameter n_d in such integrals plays the role of the shift of the dimension. Suppose that we are not interested in higher terms of the Taylor expansion in powers of $(q^2 - m_{N+1}^2)$ in (97), i.e. we need just the value at $q^2 = m_{N+1}^2$, i.e. the term with $a_{N+1} = 1$. Then the integration over x_{N+1} should be understood in the sense of Cauchy integration so that, effectively, x_{N+1} is set to zero. So, if $\hat{P}(x_1, \ldots, x_N, x_{N+1})$ is the basic polynomial calculated for this augmented vacuum configuration, the true basic polynomial $P(\underline{x})$ for our physical propagator problem is recovered via:

$$P(\underline{x}) \equiv P(x_1, \ldots, x_N) = \hat{P}(x_1, \ldots, x_N, 0). \tag{100}$$

When dealing with a general case containing n independent external momenta $q_1, \ldots, q_n$, we include all external scalar products $q_i \cdot q_j$ directly into the algebraic procedure. The formal transition to the vacuum problem—which effectively increases the apparent loop count from $h \to h + n$—can be summarized by the following systematic protocol:

1. Introduce a comprehensive set of vectors p_i, in addition to $k_i \cdot k_j$ $(i > j)$, $k_i \cdot q_j$ and also invariants generated by external momenta, i.e. the scalar products $q_i \cdot q_j$ $(i \geq j)$. The vectors p_i places loop momenta and external momenta on an equal footing:

$$p_i = \begin{cases} k_i & \text{for } i = 1, \ldots, h \\ q_{i-h} & \text{for } i = h+1, \ldots, h+n. \end{cases} \tag{101}$$

This expands the total number of independent kinematic invariants (including both internal and external scalar products) to:

$$\hat{N} = \frac{(h+n)(h+n+1)}{2}. \tag{102}$$

2. Construct a set of $\hat{N}$ linearly independent quadratic denominators matching the total number of invariants. So, introduce, in some way, the corresponding new propagators.

3. Solve the linear system:

$$\sum_{i \geq j \geq 1}^{h+n} A_r^{ij} p_i \cdot p_j = E_r \quad \text{for } r = 1, \ldots, \hat{N} \tag{103}$$

with respect to the full set of scalar products $p_i \cdot p_j$.

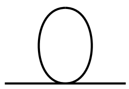
4. Compute the determinant of the symmetrized inverse matrix configuration to obtain the expanded polynomial $\hat{P}(x_1, \ldots, x_{\hat{N}})$.

5. Recover the true physical Baikov polynomial $P(\underline{x})$ by freezing the auxiliary variables corresponding to purely external invariants at their fixed kinematic values (setting the remaining auxiliary coordinates to zero):

$$P(\underline{x}) \equiv P(x_1, \ldots, x_N) = \hat{P}(x_1, \ldots, x_N, 0, \ldots, 0). \tag{104}$$

It is worth stressing that this strategy is incredibly robust. It applies not only to standard Feynman integrals with quadratic denominators, but also to more general effective field theory setups (such as HQET or NRQCD), where linear structures like heavy-quark velocities v^μ can be treated seamlessly as additional external vectors within the exact same algebraic pipeline. So it's valid for more general Feynman integrals with generic denominators, like (82).

Example 5. Again, let's focus on the tadpole:



$$= \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^a} \equiv F(a). \tag{105}$$

We have one propagator with the denominator:

$$E = k^2 - m^2 \tag{106}$$

and one kinematical invariant k^2. The equation $E = k^2$ is solved as $k^2 = E$. Therefore, the resulting basic polynomial is $P(x) = x$ and the polynomial that enters (94) is just:

$$P(x') = x + m^2. \tag{107}$$

There is one master integral $I_1 = F(1)$ given by the right-hand side of (62), so:

$$I_1 = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)} = -i\pi^{d/2} \Gamma\left(1 - \frac{d}{2}\right) (m^2)^{d/2-1}.$$

According to (94) the corresponding coefficient function is:

$$c(a) \sim \int \frac{\mathrm{d}x}{x^a} (x + m^2)^{(d-2)/2} \tag{108}$$

$$= \frac{1}{2\pi i} \oint \frac{\mathrm{d}x}{x^a} (x + m^2)^{(d-2)/2}. \tag{109}$$

We can see that at $a = 1$ we have:

$$\frac{1}{2\pi i} \oint \frac{\mathrm{d}x}{x^a} (x + m^2)^{(d-2)/2} = (x + m^2)^{(d-2)/2} \tag{110}$$

$$= (m^2)^{(d-2)/2}. \tag{111}$$

In order to satisy the normalization condition $c(1) = 1$ we normalize the coefficient function as:

$$c(a) = \frac{(m^2)^{(2-d)/2}}{2\pi i} \oint \frac{\mathrm{d}x}{x^a}(x+m^2)^{(d-2)/2} \tag{112}$$

$$= \frac{(m^2)^{(2-d)/2}}{(a-1)!} \left(\frac{\partial}{\partial x}\right)^{a-1} \left[(x+m^2)^{(d-2)/2}\right]\Bigg|_{x=0} \tag{113}$$

for $a = 1, 2, \ldots$. So, we have:

$$F(a) = c(a) I_1 \tag{114}$$

in agreement[7] with (61).

[7] It's easy to verify the generic the derivative:

$$\frac{\partial^n}{\partial x^n}(x+c)^k = k \cdot (k-1) \ldots (k-n+1)(x+c)^{k-n}$$

so in the case $n = a-1$, $c = m^2$ and $k = (d-2)/2$:

$$\left(\frac{\partial}{\partial x}\right)^{a-1} (x+m^2)^{(d-2)/2} = \left(\frac{d-2}{2}\right)\left(\frac{d-2}{2}-1\right)\ldots\left(\frac{d-2}{2}-a+1+1\right)(x+m^2)^{(d-2)/2-a+1}$$
$$= (-1)^{a-1}\left(1-\frac{d}{2}\right)_{a-1}(x+m^2)^{\frac{d}{2}-a}$$

which gives, if valutated at $x = 0$:

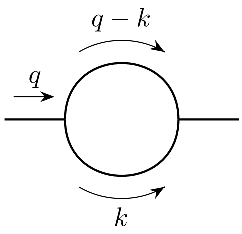
$$c(a) = \frac{(m^2)^{(2-d)/2}}{(a-1)!}\left(\frac{\partial}{\partial x}\right)^{a-1}\left[(x+m^2)^{(d-2)/2}\right]\Bigg|_{x=0}$$
$$= \frac{(m^2)^{1-\frac{d}{2}}}{(a-1)!}(-1)^{a-1}\left(1-\frac{d}{2}\right)_{a-1}(m^2)^{\frac{d}{2}-a}$$
$$= \frac{(-1)^{a-1}}{(a-1)!\,(m^2)^{a-1}}\left(1-\frac{d}{2}\right)_{a-1}. \tag{115}$$

A Some result

In this Appendix you can find some well-known result obtained via Feynman or alpha (Schwinger) representation, which are a good check for the results obtained with the Mellin-Barnes representation.

A.1 General formula for the self energy with Feynman parameters

We want to prove the formula:



$$= \int \frac{\mathrm{d}^d k}{(-k^2+m_1^2)^{\lambda_1}(-(q-k)^2+m_2^2)^{\lambda_2}} \tag{116}$$

$$\text{Putting } m_1 = m_2 = 0 \tag{117}$$

$$= \frac{i\pi^{d/2}}{(-q^2)^{\lambda_1+\lambda_2+\epsilon-2}} \frac{\Gamma(\lambda_1+\lambda_2+\epsilon-2)\Gamma(2-\epsilon-\lambda_1)\Gamma(2-\epsilon-\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(4-\lambda_1-\lambda_2-2\epsilon)}. \tag{118}$$

We can start using the standard relation for alpha parametrization (we called Schwinger parametrization):

$$\frac{1}{(-k^2+m^2)^\lambda} = \frac{i^\lambda}{\Gamma(\lambda)} \int_0^\infty \mathrm{d}\alpha\, \alpha^{\lambda-1} e^{i(k^2-m^2)\alpha} \tag{119}$$

for both the denominator:

$$\int \mathrm{d}^d k\, \frac{i^{\lambda_1+\lambda_2}}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \int_0^\infty \mathrm{d}\alpha_1 \int_0^\infty \mathrm{d}\alpha_2\, \alpha_1^{\lambda_1-1}\alpha_2^{\lambda_2-1}\, e^{i(k^2-m_1^2)\alpha_1} e^{((q-k)^2-m_2^2)\alpha_2} \tag{120}$$

changing the integral in k and in α_i we can collect the terms with k, to see:

$$\int \mathrm{d}^d k\, e^{i[k^2(\alpha_1+\alpha_2)-2q\cdot k\alpha_2]} = \int \mathrm{d}^k\, e^{i(\alpha_1+\alpha_2)\left[k^2-2q\cdot k\frac{\alpha_2}{(\alpha_1+\alpha_2)}\right]} \tag{121}$$

$$= \int \mathrm{d}^k\, e^{i(\alpha_1+\alpha_2)\left[\left(k-q\frac{\alpha_2}{\alpha_1+\alpha_2}\right)^2-q^2\frac{\alpha_2^2}{(\alpha_1+\alpha_2)^2}\right]} \tag{122}$$

$$= e^{-iq^2\frac{\alpha_2^2}{(\alpha_1+\alpha_2)}} \int \mathrm{d}^d\tilde{k}\, e^{i(\alpha_1+\alpha_2)\tilde{k}^2} \tag{123}$$

$$= e^{-iq^2\frac{\alpha_2^2}{(\alpha_1+\alpha_2)}} (\alpha_1+\alpha_2)^{-d/2}\, \pi^{d/2}\, e^{i\frac{\pi}{2}\left(1-\frac{d}{2}\right)} \tag{124}$$

where the term $e^{i\frac{\pi}{2}\left(1-\frac{d}{2}\right)}$ is a Minkowski phase. So, putting everything together:

$$\frac{i^{\lambda_1+\lambda_2+1-\frac{d}{2}}\pi^{d/2}}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \int \mathrm{d}\alpha_1 \int \mathrm{d}\alpha_2\, \frac{\alpha_1^{\lambda_1-1}\alpha_2^{\lambda_2-1}}{(\alpha_1+\alpha_2)^{d/2}} \exp\left\{iq^2\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2} - i(m_1^2\alpha_1+m_2^2\alpha_2)\right\} \tag{125}$$

we can do the change of variables:

$$\eta = \alpha_1+\alpha_2 \quad , \quad \xi = \frac{\alpha_1}{\alpha_1+\alpha_2} \quad \Longrightarrow \quad \mathrm{d}\alpha_1\mathrm{d}\alpha_2 = \eta\mathrm{d}\eta\mathrm{d}\xi \tag{126}$$

that gives us:

$$\alpha_1 = \eta\xi \quad , \quad \alpha_2 = \eta(1-\xi) \tag{127}$$

to write the result:

$$\frac{i^{\lambda_1+\lambda_2+1-\frac{d}{2}}\pi^{d/2}}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \int \mathrm{d}\xi\, \xi^{\lambda_1-1}(1-\xi)^{\lambda_2-1} \int_0^\infty \mathrm{d}\eta\, \eta^{\lambda_1+\lambda_2-\frac{d}{2}-1} \exp\left\{i\eta\left[q^2\xi(1-\xi)-(m_1^2\xi+(1-\xi)m_2^2)\right]\right\} \tag{128}$$

noticing in the last integral a gamma function, we have:

$$\int \mathrm{d}\eta\, \eta^{\alpha-1} e^{-i\eta\Delta} = \frac{\Gamma(\alpha)}{(i\Delta)^{\alpha}} \qquad , \quad \Delta = m_1^2\xi + (1-\xi)m_2^2 - q^2\xi(1-\xi)$$

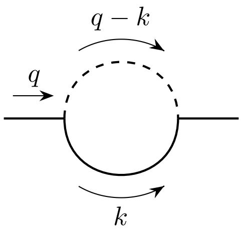
so we finally obtain:

$$\int \frac{\mathrm{d}^d k}{(-k^2+m_1^2)^{\lambda_1}(-(q-k)^2+m_2^2)^{\lambda_2}} = \\ = \frac{i\pi^{d/2}}{\Gamma(\lambda_1)\Gamma(\lambda_2)}\Gamma\left(\lambda_1+\lambda_2-\frac{d}{2}-1\right)\int_0^1 \mathrm{d}\xi\, \frac{\xi^{\lambda_1-1}(1-\xi)^{\lambda_2-1}}{\left[m_1^2\xi+(1-\xi)m_2^2-q^2\xi(1-\xi)\right]^{\lambda_1+\lambda_2-d/2}}. \tag{129}$$

If we suppose that the masses are zero the integral over ξ can be evaluated in terms of gamma function and we obain (118).

A.2 One-Loop self energy with Feynman parameters

We want to prove:



$$= \int \frac{\mathrm{d}^k}{(k^2-m^2)^{a_1}[(q-k)^2]^{a_2}} \tag{130}$$

Putting $a_1 = 2$ and $a_2 = 1$ (131)

$$= \int \frac{\mathrm{d}^d k}{(k^2-m^2)^2(q-k)^2}. \tag{132}$$

we can use again the Feynman parameter to get (doing following the same path of section §A.1):

$$\int \frac{\mathrm{d}^d k}{(k^2-m^2)^2(q-k)^2} = \int \mathrm{d}^d k \int_0^1 \frac{\xi \mathrm{d}\xi}{\left[(k^2-m^2)\xi+(1-\xi)(q-k)^2\right]^3} \tag{133}$$

and again, changing the order of integration over ξ and k, and using the expession for the tadpole:

$$\int \frac{\mathrm{d}^d k}{(-k^2+m^2)^{\lambda}} = i\pi^{d/2}\frac{\Gamma(\lambda+\epsilon-2)}{\Gamma(\lambda)}\frac{1}{(m^2)^{\lambda+\epsilon-2}}$$

which is equivalent to (61), we can obtain:

$$\int \frac{\mathrm{d}^d k}{(k^2-m^2)^2(q-k)^2} = -i\pi^{d/2}\Gamma(1+\epsilon)\int_0^1 \frac{\mathrm{d}\xi\, \xi^{-\epsilon}}{\left[m^2-q^2(1-\epsilon)\right]^{1+\epsilon}} \tag{134}$$

which can be easily evaluated at $d = 4$:

$$\int \frac{\mathrm{d}^4 k}{(k^2-m^2)^2(q-k)^2} = \frac{i\pi^2}{q^2}\ln\left(1-\frac{q^2}{m^2}\right). \tag{135}$$

In principle, any given Feynman integral $F(a_1, a_2, d)$ with concrete numbers a_1 and a_2 can similarly be evaluated by Feynman parameters. In particular, $F(1, 1, d)$ reduces to:

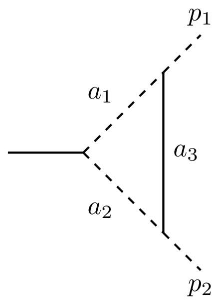
$$i\pi^{d/2}\Gamma(\epsilon)\int_0^1 \frac{\mathrm{d}\xi\, \xi^{-\epsilon}}{\left[m^2-q^2(1-\xi)\right]^{\epsilon}} \tag{136}$$

where there is an ultraviolet (UV) divergence which manifests itself in the first pole of the function $\Gamma(\epsilon)$, i.e. at $d = 4$. The integral can be evaluated in expansion in a Laurent series in ϵ, for example, up to ϵ^0:

$$i\pi^{d/2}e^{-\gamma_E\epsilon}\left[\frac{1}{\epsilon}-\ln m^2+2-\left(1-\frac{m^2}{q^2}\right)\ln\left(1-\frac{q^2}{m^2}\right)+\mathcal{O}(\epsilon)\right]. \tag{137}$$

A.3 Triangle in alpha representation

We want to solve:



$$= I_2^{(1)[d]}(q^2, m^2; a_1, a_2, a_3, d) \tag{138}$$

$$= \int \frac{\mathrm{d}^d k}{(k^2 - 2p_1 \cdot k)^{a_1}\,(k^2 - 2p_2 \cdot k)^{a_2}\,(k^2 - m^2)^{a_3}}. \tag{139}$$

We can use the alpha representation and then do the change:

$$\alpha_1 = \xi_1 \eta \quad , \quad \alpha_2 = \xi_2 \eta \quad , \quad \alpha_3 = (1 - \xi_1 - \xi_2)\eta \tag{140}$$

integrate over η and obtain:

$$I_2^{(1)[d]}(q^2, m^2; a_1, a_2, a_3, d) = \\ = \frac{i\pi^{d/2}(-1)^{a_1+a_2+a_3}\Gamma(a + \epsilon - 2)}{\prod_l \Gamma(a_l)} \int_0^1 \mathrm{d}\xi_1 \int_0^{1-\xi_1} \mathrm{d}\xi_2\, \frac{\xi_1^{a_1-1}\xi_2^{a_2-1}(1 - \xi_1 - \xi_2)^{a_3-1}}{[-q^2\xi_1\xi_2 + m^2(1 - \xi_1 - \xi_2)]^{a+\epsilon-2}} \tag{141}$$

with $a = a_1 + a_2 + a_3$. This can be a reasonable starting point for the evaluation of integrals with any given indices a_i. Let us evaluate the integral with $a_1 = a_2 = a_3 = 1$ at $d = 4$. Then the integral is finite:

$$I_2^{(1)[d]}(q^2, m^2; 1, 1, 1, 4) = -i\pi^2 \int_0^1 \mathrm{d}\xi_1 \int_0^{1-\xi_1} \frac{\mathrm{d}\xi_2}{-q^2\xi_1\xi_2 + m^2(1 - \xi_1 - \xi_2)} \tag{142}$$

which gives the result:

$$I_2^{(1)[d]}(q^2, m^2; 1, 1, 1, 4) = \frac{i\pi^2}{(-q^2)} \left[-\frac{\pi^2}{3} + \mathrm{Li}_2(x) - \frac{1}{2}\ln^2\left(\frac{m^2}{-q^2}\right) + \ln\left(\frac{m^2}{-q^2}\right)\ln\left(1 - \frac{m^2}{-q^2}\right)\right]. \tag{143}$$

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