Notes from the course on:

Advanced Quantum Fields Theory

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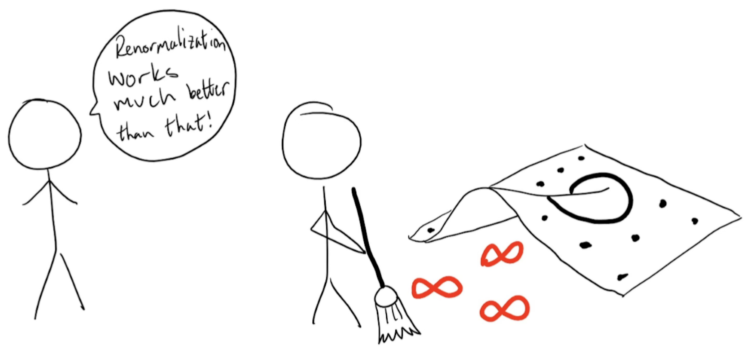
A.A. 2025-2026



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Let no one ignorant of geometry
enter my door.

Plato

Preface

In this document I aim to collect my notes based on the material from the course "**Advanced Quantum Fields Theory**", taught by Prof. Simon Badgerand attended at the *University of Turin* in the academic year 2025-2026. During the course, several books were recommended (listed in the Bibliography), and I will try to indicate the corresponding references at the beginning of each chapter.

These notes are a rewritten version of the notes I took during the lectures, so the main source is the material presented by the professor. I disclaim all responsibility and refer you to my personal page to read more about these notes and the fact that they don't have neither academic nor instructional nature.

These notes should clearly be understood as personal, neatly rewritten lecture notes. Any oversights, mistakes, or inaccuracies are due to my own limitations. Moreover, I wrote these notes mainly to "explain" the subject to myself, so some sections may appear overly detailed or, conversely, too superficial depending on the reader. In any case, I hope they may still be useful to someone. I also hope that I have managed to produce a clear and well-structured document.

A collection of my notes is available on my personal GitHub page: gCembalo.github.io.

Any error or typo can be reported to my personal email:
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Contents

Chapter 1

Intoduction

In this course we want to develop the formalism of Quantum Field Theory to describe:

- *Loop computations*: quantum corrections in perturbation theory.
- The origin of *UV divergences* and the method of *renormalization.*
- *Renormalization scale indipendence* and the *renormalization group.*
- *Gauge invariance* and *unitarity* for non-abelian gauge theories.

The course structure is: a brief review of QFT, the $\lambda\phi^3$ theory, we will study the loop computation methods, we will see how to renormalize the QED and QCD to develop the formalism for the renormalization group, but the last part of the course is unitarity, "cuts" and non-abelian gauge invariance.

As presequisites we need to know how to derive Feynman rules and tree-level ampludes, we must be familiar with LSZ formalism and the correction between Green's functions and S-matrix elements and we must have basic knowledge of abelian, and non-abelian, gauge theory (QED and QCD).

Basic QFT history. In 1927 P. A. M. Dirac find a *quantum theory of electrons*, but with one big problem, the electron self-energy is infinity. Between 1947 and 1949 Feynman, Schwinger and Tomonaga develop the *renormalization of QED*. We can think that in that moment they build the QFT.

Note that was also contributions from Kramers, Bethe and Dyson.

Later on the physicist develop the Yang-Mills theory, the QCD, Weinberg (and other in '67) develop the Standard Model and the EW unification, also 'T Hooft and Veltman in '72 contributed to EW renormalization.

Between 1972 and 1975 has been developed by Wilson, based on preceding work by Stwekelber, Peterman, Gell-Mann, Low and other, the renormalization group theory.

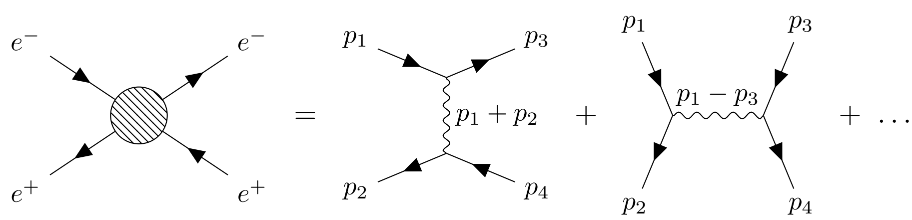
Sometimes the philosophy behind the renormalization process is a little bit confusing, but in the following sections we will see a brief overview to try to understand what we will have to do in order to "fix" the Quantum Field Theory. The (following) material in this chapter are inspired from *ZAP physics*' Youtube video and it's not part of the exam.

In the video he shows the problem of divergences, he make an easy analogy about the renormalization scheme from another problem and then he shows how we can redefine the physical quantities and make our theories useful. I took the image in the cover from his video.

It can be usefull to see the notes from MSc course *Phenomenology of fundamental interaction*, where we use some of the concepts (regularization and renormalization) applied to Fermi theory and QCD.

1.1 Overview

We already know, from previous courses, that we can do perturbation theory of a particular process. For example, we can compute the *Bhabha scattering*:



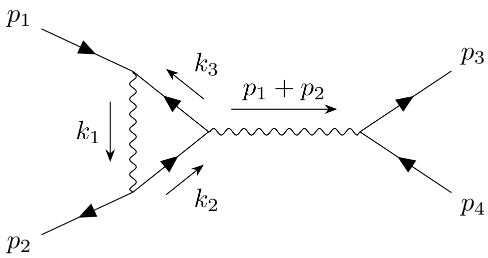
The problem, that arise quite easily in calculation, is that some of the diagrams *diverges*. Some of this infinities are nothing to worry about because they cancel out considering other related processes, however there are some infinities left over and seem to cause massive problem. For example, if we compute the scattering cross section for a particular process, but it contains one (or more) feynman diagram that diverges, it's not a good thing have an infinity probability to observe a process.

We can cure the theory to have a meaningful object in our hands, we need to analyze the theory of **renormalization**. In particular we will study the **perturbative renormalization**.

The renormalization is a little bit touchy at first sight, so it helps a lot see an example from another math argument. But first let's see from where this divergences come from.

1.2 The origin of the problem

Consider the Feynman diagram:



which is a second order correction to Bhabha scattering. We know that in each vertex we have to conserve the momentum:

$$\begin{cases} p_1 + k_3 = k_1 \\ p_2 + k_1 = k_2 \\ k_2 = k_3 + p_3 + p_4 = k_3 + p_1 + p_2 \end{cases} \tag{1.2.1}$$

And we also know that not all the momentum of internal lines are fixed; in facts for each loop (closed) we have in the diagram, we get an unspecified momentum, and since we have to add up all possibilities for internal state, we have to integrate over k.

The integral is the reason why we get infinities and they can arise in different region (of k). In particular we say:

- **IR divergences**: arise from region where $k \to 0$.
- **UV divergences**: arise from region where $k \to \infty$.

This divergences don't arise in every diagram with a loop in it, but it depends on the number (and type) of propagator (internal lines). We will talk during the course about the *superficial degree of divergence* ω, which help us to understand if we have a divergence.

1.3 Singular perturbation theory

Let's see an example, known as *singular perturbation theory*, to understand the procedure we will follow during the renormalization process.

Consider the equation:

$$\epsilon x^2 - 2x + 1 = 0 \quad \text{with} \quad \epsilon \ll 1 \tag{1.3.1}$$

We want to solve for x but without use the quadratic formula. We can't solve it exactly, but we can do it perturbatively on ϵ. We guess:

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots \tag{1.3.2}$$

We can put $x(\epsilon)$ in the original equation and see:

$$\epsilon(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots)^2 - 2(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots) + 1 = 0 \tag{1.3.3}$$

$$\Rightarrow (-2x_0 + 1) + \epsilon(x_0^2 - 2x_1) + \epsilon^2(2x_0 x_1 - 2x_2) + \dots = 0 \tag{1.3.4}$$

So, we can solve this equation order by order in ϵ:

$$\Rightarrow x_0 = \frac{1}{2} \quad , \quad x_1 = \frac{1}{8} \quad , \quad x_2 = \frac{1}{16} \quad , \quad \dots \tag{1.3.5}$$

But we find just one solution and we know that the root of the original equation are two.

What happened was that we take the limit for $\epsilon \to 0$ in $\epsilon x^2 - 2x + 1 = 0$, so we get a linear equation with only one root.

But there is another way to get an equation, quadratic, that in $\epsilon \to 0$ limit gives us the two root.

We have to redefine the variable $x = \delta(\epsilon)y$, where $\delta(\epsilon)$ is a parameter that depends on ϵ and which we can fix in order to solve the problematic behavior. We will solve the equation for y instead of x.

Note that $\delta(\epsilon)$ is completely arbitrary. We have the equation:

$$\epsilon\delta(\epsilon)^2 y^2 - 2\delta(\epsilon)y + 1 = 0 \tag{1.3.6}$$

where a good choice we can make is $\delta(\epsilon) = \frac{1}{\epsilon}$ and then multiply all by ϵ:

$$y^2 - 2y + \epsilon = 0 \tag{1.3.7}$$

which is perfectly well behaved for $\epsilon \to 0$. Now, we can do the same thing as before, so write for y a perturbative solution in ϵ:

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots \tag{1.3.8}$$

and solve order by order:

$$\Rightarrow (y_0^2 - 2y_0) + \epsilon(2y_0 y_1 - 2y_1 + 1) + \epsilon^2(y_1^2 + 2y_0 y_2 - 2y_2) + \dots = 0 \tag{1.3.9}$$

$$\Rightarrow y^{(1)} = \frac{\epsilon}{2} + \frac{\epsilon^2}{8} + \dots \quad , \quad y^{(2)} = 2 - \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \dots \tag{1.3.10}$$

So, in this way, we recover our second root and to get the solution in x we just multiply the roots by $\delta(\epsilon)$:

$$x^{(1)} = \frac{1}{2} + \frac{\epsilon}{8} + \dots \quad , \quad x^{(2)} = \frac{2}{\epsilon} - \frac{1}{2} - \frac{\epsilon}{8} + \dots \tag{1.3.11}$$

For the renormalization of QFT we will follow a similar scheme as the previous example. The procedure was:

1 Re-define variable in terms of arbitrary scale parameter.

2 Fix by hand behavior of the parameter in order to eliminate artificially the problematic behavior.

3 Solve well behaved problem and factor back in the scale parameters to get our true solution.

One more thing. The introduction of $\delta(\epsilon)$ was necessary to solve the equation, but the choice $\delta(\epsilon) = 1/\epsilon$ was completely arbitrary and we could define it adding a finite constant a:

$$\Rightarrow \delta = \delta(\epsilon, a) \quad \Rightarrow \quad y = y(\epsilon, a) \tag{1.3.12}$$

but a doesn't appear in our original problem, so when we go back to x, it cannot depend on a. We can write a differential equation about it:

$$\frac{dx}{da} = \frac{d(\delta y)}{da} = \frac{d\delta}{da} y + \delta \frac{dy}{da} \equiv 0 \tag{1.3.13}$$

$$\Rightarrow \boxed{\delta \frac{dy}{da} = -\frac{d\delta}{da} y} \tag{1.3.14}$$

Which have to be satisfied at any order in perturbation theory.

1.4 Renormalization in particle physics

Now, finally talk about particle physics. We know that in our theory we have some parameter not fixed by the theory itself, like couplings $(g \dots)$, masses (m) and field normalizations. We usually call these un-fixed parameters *bare parameters.* Some non technical talks are present in chapter §32 of Lancaster and Bludell Let no one ignorant of geometry enter my door.Lancaster.

1.4.1 What are the physical quantities?

We've already seen in MSc course *Foundation of QFT* that the physical quantities are not the ones we see in the lagrangian $\mathcal{L}$, bet we need to dress them with interaction. We saw that we needed to rescale the propagator of the scalar theory (but this is valid in general) with a factor Z. The fact that the quantities in $\mathcal{L}$ is not the physical quantities is a huge problem, because this means that we did perturbation theory about wrong masses and coupling constants.

However, when we will analize the problem in QFT we won't talk about bare parameters or physical ones, instead we will talk about divergences. Indeed, doing the well-known perturbation theory, with bare parameters, we will find that some Feynman diagrams, in particular those with loops, diverges in some regions for the loop momentum k. If we identify those divergences to the parameters (masses and coupling constants) that we are using, so the bare ones, we can state that the *bare parameters contains some divergences*, for this reason they are not the properties that we see in laboratory, *and we need to avoid these infinities to have a physical theory.*

The way we fix the theory, cancelling the infinities, is via counterterms, that contains the same bare parameter's infinities. The counterterms are some new terms that we need to add to the lagrangian of the theory, and obviously it add some new Feynman rules. We will talk a little bit of counterterms in the next subsection.

The main thing we need to understand is that adding the counterterms to the theory we are not modifing the physics behind, but we are just re-organizing the math to obtain physiscal quantities. Indeed, the new fields, masses and coupling constants that we obtain are the real quantities that we see in the experiments.

The first question that arise is: once we reabsorbed the divergences in the bare parameters, with new terms in $\mathcal{L}$, at a specific order of perturbation theory, we could have new possible divergences at higher order, so we are again at the starting point, but without the bare parameters that we want to use to cure the theory. The answer, unfortunately, sometimes is that there are some cases where we can absorbe the divergences of bare parameters with some counterterms, but doing that we introduce in the theory some new infinities, at higher order, and we have to repeat the process again, but adding some pathological therms every time. One example of these non-renormalizable theory are the Fermi theory.

Fortunately not all the lagrangian implies a never ending renormalization process, and we can cure a lot of theory. In this kind of theory, that we call *renormalizable*, we can absorbe all the divergences, at every order in perturbation theory, only with the bare parameters in $\mathcal{L}$.

The dimensionality of the coupling constants will contributes to the renormalizable or not of the theory.

1.4.2 The counterterms

During the renormalization process we rescale these parameters:

$$g_0 = Z_g g_1 \qquad m_0 = Z_m m_1 \qquad \phi_0 = Z_\phi^{1/2}\phi_1 \tag{1.4.1}$$

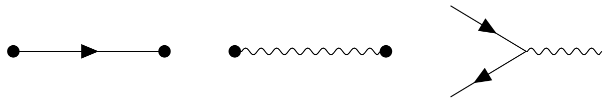
and call g_0, m_0, ϕ_0 the bare parameters, Z_g, Z_m, Z_ϕ the *renormalization constants* and g_1, m_1, ϕ_1 the *dressed parameters.*

Typically we write:

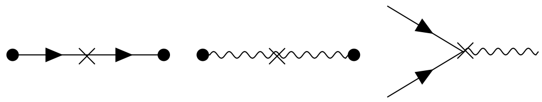
$$Z_g = 1 + \delta Z_g \qquad Z_m = 1 + \delta Z_m \qquad Z_\phi = 1 + \delta Z_\phi \tag{1.4.2}$$

where we call δZ *counter-terms.*

With this procedure we can keep the Feynman rules, but with bare parameters replaced with the renormalized counterparts:

 (1.4.3)

However we have to add new Feynman rules, we didn't have before, related to counter term:



$\downarrow$ $\qquad\qquad$ $\downarrow$ $\qquad\qquad$ $\downarrow$

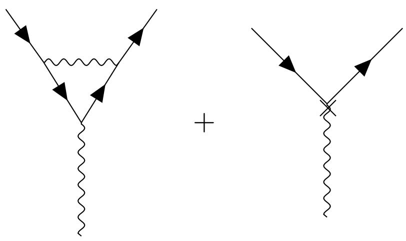
$\delta Z_\psi, \delta Z_{m_\psi}$ $\qquad$ δZ_A $\qquad$ δZ_g

> **Remember.** We have the same theory as before, but we've just shuffled things around to make it look slightly different (and solve some problems).

Note. The counter terms have the same role of $\delta(\epsilon)$ in the example.

Now, we have to fix these counter term to eliminate UV divergences (just like we fixed δ to eliminate singular behavior in y^2 term).

When we consider a process with specific external lines, we can build the similar for the counter terms. For example:

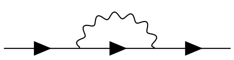
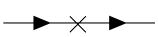


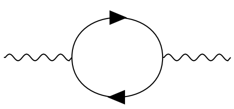
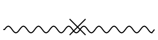
and if we require that this sum is *finite* we fix δZ_g (to one-loop order). The relation:

$$\delta Z = (\text{UV divergence}) + (\text{finite}) \tag{1.4.4}$$

gives us the renormalization scheme (MS, $\overline{\text{MS}}$ or on-shell scheme).

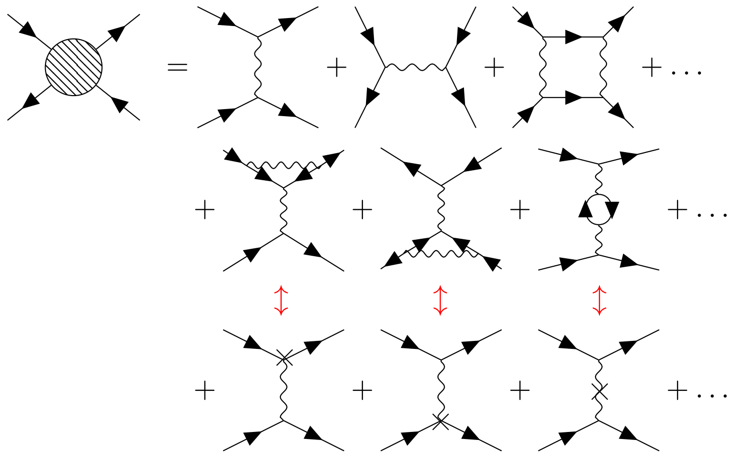
Then we can do the same thing with the loop correction to the propagator:

 +  = finite $\longrightarrow$ fixes $\delta Z_\psi, \delta Z_{m_\psi}$ (1.4.5)

 +  = finite $\longrightarrow$ fixes δZ_A (1.4.6)

So, at this point we have fixed all the counter-terms, so that they fix all our divergences in our theory.

Coming back to Bhabha scattering we can do the calculation as before but using the dressed parameters:



(1.4.7)

Where we included all the diagrams with the counter term (up to a given order). It's easy in this example that we can pair the divergent diagram with the correspondent counter term, and the sum gives us (because we impose that) a finite quantity. So, at the end we get a finite amplitude for the process.

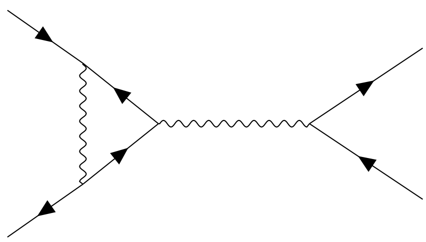
So far what we did is:

1. Re-scale unspecified parameters in theory ($g_0 = (1 + \delta Z_g)g_1 \dots$)
2. Fix counter-terms to cancel specific UV divergences in the theory
3. "Multiply" by scale factors by including Feynman diagrams which include counterterm insertions.

With this procedure we can make the scattering amplitude finite.

1.4.3 The renormalization group

Unfortunately this is not the end of the story (we did a course for that). When we have a divergent diagram, such as:



$= \infty$

we still have important finite behaviors in it we need to consider (also the structure of the divergence is important). So we have to regulate the divergence somehow to force to give us a finite result. What we want to do is separate the finite part and the infinite part, to keep the physical one and reabsorb the other.

This **regularization** can be done in different ways, the most popular are:

- Momentum cut-off Λ.
- Dimensional regularization: we go in arbitrary dimension $d = 4 - 2\epsilon$ and take $\epsilon \to 0$ at the end of calculation.

With these methods we introduce an energy scale, which is artificial.

So, to regulate integrals, and find the counter terms, we have to (somehow) introduce a new artificial energy scale μ, which, since it doesn't appears in the original theory, the bare parameters cannot depend on μ (for the same reason x couldn't depend on a in the example). As before we get a set of differential equations:

$$\boxed{\frac{dg}{d\log\mu} = -g\frac{1}{Z_g}\frac{dZ_g}{d\log\mu} \quad , \quad \frac{dm}{d\log\mu} = -m\frac{1}{Z_m}\frac{dZ_m}{d\log\mu}} \tag{1.4.8}$$

This set of differential equations are called *renormalization group equations*. These equations guarantee that physical quantities are completely independent from the artificial energy scale.

Now we have that the renormalized parameters, in general, depend on energy scale μ, and when we do perturbation series, now in terms of g (not g_0), can happen that even if g is small at some scale μ_0, at another scale μ_1, g may become large and the perturbation theory break down in this region. This is what happens for low-energy QCD.

In addition, to fix the UV divergence of our theory we can work with the counter terms, as we saw. But it's possible that some infinities remain in the theory and that we have to include new interaction (new vertices). This hasn't to bother us because what we are saying adding this new interaction is that in the energy region where we have the infinity, there are some process that arise only there.

But we can have *renormalizable theory*, where we need to add a finite number of new interaction, and *non-renormalizable theory* where we need to add infinite new interaction.

Note. Adding new interaction we can build new counter-term that we can use to regulate the theory.

The non-renormalizable theory are not useless, but we can define them as **effective field theory**, so as theory valid up to a scale Λ (where the number of counter-terms we need to include are finite). For example the *Fermi theory* for weak interactions.

To put it in a nutshell: bare parameters are unobservable and divergent, but they exactly cancel out the loop divergences, leaving behind the finite, physical values we actually measure in the laboratory.

Unfortunately, we cannot mathematically subtract ∞ from ∞ and get a meaningful answer. Regularization (like dimensional regularization or Pauli-Villars) acts as a mathematical scaffold. It temporarily tames the infinities by introducing a parameter—like ϵ in $d = 4 - \epsilon$ dimensions, and this allows us to isolate the divergence explicitly (typically as poles like $1/\epsilon$).

Once the divergence is isolated, we split our unobservable bare parameters (ϕ_0, m_0, g_0) into renormalized parameters (ϕ, m, g) and counterterms $(\delta_\phi, \delta_m, \delta_g)$. We can write:

$$\mathcal{L}_{\text{bare}} = \mathcal{L}_{\text{renormalized}} + \mathcal{L}_{\text{counterterms}} \tag{1.4.9}$$

The Feynman rules generated by $\mathcal{L}_{\text{counterterms}}$ give rise to new vertices (counterterm diagrams) that we add to our loop diagrams.

As we said, there is one detail to add to our understanding. While the counterterm must cancel the divergent piece, we have some freedom in how much of the *finite* piece it also absorbs. This choice is called a *renormalization scheme* and we will see some of them.

So, while regularization *identifies* the divergence, an experimental measurement or a convention (the scheme) is what ultimately fixes the exact value of the counterterm.

At the end we need to request that our observable don't depends on the choise we made in our renormalization process.

1.5 Procedure to cure the theory

In Iliopoulos et al. Let no one ignorant of geometry enter my door.Iliopoulos they show that the divergences in the theory arise because we applied incorrectly the quantisation rules. They tell us that we need to follow a precise procedure, which we already saw in this chapter, to renormalize, so make physically acceptable, the theory:

1. The *power counting*, which tells us, not only if a theory is renormalizable, but either how many parameters we need to cure the infinities.

2. The *regularization*, which is a prescription to make every Feynman diagram finite at the price of introducing a new parameter in the theory. We will see that there are several method to make this.

3. The *renormalization*, which is the mathematical procedure to remove the regularization parameter and determine the value of counterterm, which modify the fields' coefficients in the lagrangian.

Chapter 2

Loop correction in scalar Quantum Field Theory

In this introductory chapter we review the formalism for perturbative scalar field theory. You can see my lecture notes from the other two QFT course.

Note. In this chapter we will represent the Feynman diagrams for scalar fields with solid lines, and not with dashed.

2.1 Basic formalism

The lagrangian density for a real scalar field, in $4d$ Minkowski spacetime using diagonal metric:

$$g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \tag{2.1.1}$$

with an arbitrary potential $V(\phi)$ is:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{2.1.2}$$

where $\phi = \phi(x^\mu)$ con x^μ is a 4-vector in Minkowski spacetime. We will take the convention $\hbar = c = 1$ (you can read the Appendix A) throughout. The classical equation of motion (Euler-Lagrange equations) are:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \quad\Longrightarrow\quad \partial_\mu\partial^\mu\phi + \frac{\partial V}{\partial\phi} = 0. \tag{2.1.3}$$

As we learn, we can quantize the theory by imposing *canonical commutation relations* between equal time fields and canonical momenta $\pi(x^\mu)$:

$$\pi(x^\mu) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} \tag{2.1.4}$$

that we choose:

$$\big[\phi(t,\vec{x}),\phi(t,\vec{x}')\big]=0 \tag{2.1.5}$$
$$\big[\pi(t,\vec{x}),\pi(t,\vec{x}')\big]=0 \tag{2.1.6}$$
$$\big[\pi(t,\vec{x}),\phi(t,\vec{x}')\big]=-i\delta^{(3)}(\vec{x}-\vec{x}'). \tag{2.1.7}$$

The hamiltonian is then:

$$H=\int \mathrm{d}^3x\left(\frac{\pi^2}{2}+\frac{(\vec{\partial}\phi)^2}{2}+V(\phi)\right) \tag{2.1.8}$$

with:

$$(\vec{\partial}\phi)^2=(\partial_i\phi)(\partial^i\phi)=\sum_{i=1}^{3}(\partial_i\phi)^2. \tag{2.1.9}$$

Doing so, the field ϕ is now interpreted as an operator on Hilbert space:

$$\phi \quad\longrightarrow\quad \hat{\phi} \tag{2.1.10}$$

satisfying the Heisenberg equation of motion:

$$i\frac{\partial\hat{\phi}}{\partial t}=\Big[\hat{\phi},H\Big]. \tag{2.1.11}$$

Remember, we assume that, not only exist a Poincaré invariant vacuum state, but also that the solution of the theory ($\hat{\phi}$ and the Fock space) may be obtain using the n-point correlation functions between fields:

$$G_n(x_1,\ldots,x_n)=\langle 0|\,T\left[\phi(x_1),\ldots,\phi(x_n)\right]|0\rangle \tag{2.1.12}$$

where T represents time ordering. For the two point function the equations of motion imply:

$$\frac{\partial}{\partial x_1^{\mu}}\frac{\partial}{\partial x_{1\mu}}G_2(x_1,x_2)+\langle 0|\,T\left[\left(\frac{\partial V}{\partial\phi}\right)(x_1)\,\phi(x_2)\right]|0\rangle=-i\delta^{(4)}(x_2-x_1) \tag{2.1.13}$$

$$\Box_{x_1}G_2(x_1,x_2)+\langle 0|\,T\left[V'(x_1)\phi(x_2)\right]|0\rangle=-i\delta^{(4)}(x_2-x_1) \tag{2.1.14}$$

or for the general case:

$$\begin{aligned}\Box_{x_1}G_n(x_1,\ldots,x_n)+\langle 0|\,T\left[V'(x_1)\ldots\phi(x_n)\right]|0\rangle&=\\&=-i\delta^{(4)}(x_2-x_1)G_{n-1}(x_2,\ldots,x_n).\end{aligned} \tag{2.1.15}$$

2.1.1 The generating functional for correlation functions

Let us remind ourselves of the solution for G_2 in the free theory:

$$G_2(x_1, x_2) = \langle 0| T [\phi(x_1)\phi(x_2)] |0\rangle \tag{2.1.16}$$
$$= D_F(x_1 - x_2) \tag{2.1.17}$$
$$= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{ik(x_2 - x_1)}}{k^2 - m^2 + i\emptyset^+} \tag{2.1.18}$$

where $D_F(x_1 - x_2)$ indicates this is the *Feynman propagator* for a particle of mass m (using $V(\phi) = \frac{1}{2}m^2\phi^2$) and we have introduced a small positive immaginary term $\emptyset^+$ in the denominator to resolve the poles on the real axis.

For the interacting theory we may write (thanks its semplicity) the solution using the *path integral* formalism:

$$\langle 0| T [\phi(x_1)\phi(x_2)] |0\rangle = \lim_{t\to\infty} \frac{\int \mathcal{D}_\phi \, \phi(x_1)\phi(x_2) \, \exp\Big\{ i \int_{-t}^{t} \mathrm{d}^4 x \, \mathcal{L} \Big\}}{\int \mathcal{D}_\phi \, \exp\Big\{ i \int_{-t}^{t} \mathrm{d}^4 x \, \mathcal{L} \Big\}} \tag{2.1.19}$$

which we can solve using a generating functional $Z[J]$ with a source term J:

$$Z[J] = \int \mathcal{D}_\phi \, \exp\Big\{ i \int \mathrm{d}^4 x \, (\mathcal{L} + J(x)\phi(x)) \Big\} \tag{2.1.20}$$

so the correlation function, $G_2(x_1, x_2)$, is obtained via functional derivatives of $Z[J]$ with respect to the source term J:

$$G_2(x_1, x_2) = \langle 0| T [\phi(x_1)\phi(x_2)] |0\rangle \tag{2.1.21}$$
$$= \frac{1}{Z[0]} \left(-i \frac{\delta}{\delta J(x_1)} \right) \left(-i \frac{\delta}{\delta J(x_2)} \right) Z[J] \bigg|_{J=0} . \tag{2.1.22}$$

2.2 Two point correlation function for ϕ^3 theory in position and momentum space

Now, let's see an example that helps us to identify the problem which arise in QFT. We will only consider perturbative solutions here. Let us specify a particular interaction, the ϕ^3 theory, in the coordinates and momentum space. We take:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \tag{2.2.1}$$
$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \tag{2.2.2}$$
$$\mathcal{L}_I = -\frac{\lambda}{3!}\phi^3 . \tag{2.2.3}$$

We may expand $Z[J]$ as follows:

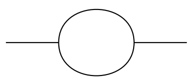
$$
\begin{aligned}
Z[J] &= \int \mathcal{D}_\phi \, \exp\Big\{ i \int \mathrm{d}^4x \, (\mathcal{L}_0 + \mathcal{L}_I) + J\phi \Big\} && (2.2.4)\\
&= \int \mathcal{D}_\phi \, \exp\Big\{ i \int \mathrm{d}^4x \, \mathcal{L}_I \Big\} \, \exp\Big\{ i \int \mathrm{d}^4x \, \mathcal{L}_0 + J\phi \Big\} && (2.2.5)\\
&= \int \mathcal{D}_\phi \, \exp\{ iS_I[\phi] \} \, \exp\Big\{ i \int \mathrm{d}^4x \, \mathcal{L}_0 + J\phi \Big\} && (2.2.6)\\
&= \int \mathcal{D}_\phi \sum_{n=0}^{\infty} \frac{1}{n!} \, (iS_I[\phi])^n \, \exp\Big\{ i \int \mathrm{d}^4x \, \mathcal{L}_0 + J\phi \Big\}. && (2.2.7)
\end{aligned}
$$

For the free theory one can derive the 2-point correlation function from the relation:

$$
\begin{aligned}
Z_0[J] = Z_0[0] \exp\Big\{ -\frac{1}{2} \int \mathrm{d}^4x \, \mathrm{d}^4y \, J(x) D_F(x-y) J(y) \Big\} && (2.2.8)\\
\implies \quad G_{2,0}(x_1, x_2) = D_F(x_1 - x_2) && (2.2.9)
\end{aligned}
$$

so, to solve the interacting theory we can expand around the free propagator:

$$
\begin{aligned}
&\frac{1}{Z_0[0]} \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \Bigg(1 + \underbrace{\frac{i\lambda}{6} \int \mathrm{d}^4z \left(\frac{\delta}{\delta J(z_1)} \right)^3}_{=0} + \\
&\quad + \left(\frac{i\lambda}{6} \right)^2 \int \mathrm{d}^4z_1 \mathrm{d}^4z_2 \left(\frac{\delta}{\delta J(z_1)} \right)^3 \left(\frac{\delta}{\delta J(z_2)} \right)^3 + \dots \Bigg) Z_0[J] \Bigg|_{J=0} && (2.2.10)\\
&= D_F(x_1 - x_2) - \\
&\quad - \frac{\lambda^2}{2} \int \mathrm{d}^4z_1 \mathrm{d}^4z_2 \, D_F(x_1 - z_1) D_F(z_1 - z_2)^2 D_F(z_2 - x_2) + \\
&\quad + \mathcal{O}(\lambda^4) && (2.2.11)
\end{aligned}
$$

$=$ $+$  $+ \ \mathcal{O}(\lambda^4)$

where we didn't draw the disconnected diagrams.

We already have an issue with this expression since the point $z_1 = z_2$ causes a singularity. The divergence is simpler to see in momentum space, so we take the Fourier transform of $G_2(x_1, x_2)$ (recalling the espression for $D_F(x_1 - x_2)$ (2.1.18)) to see:

$$
\begin{aligned}
&D_F(x_1 - z_1) D_F(z_1 - z_2)^2 D_F(z_2 - x_2) = \\
&\quad = \int \left(\prod \frac{\mathrm{d}^4k_i}{(2\pi)^4} \right) \frac{e^{ik_1(z_1 - x_1)} e^{ik_2(z_2 - x_1)} e^{ik_3(z_2 - z_1)} e^{ik_4(z_2 - x_2)}}{\prod_{i=1}^4 (k_i^2 - m^2 + i\emptyset^+)} && (2.2.12)
\end{aligned}
$$

where we can rewrite the argument of the exponential to collect on z_1 and z_2:

$$\exp\{iz_1(k_1 - k_2 - k_3) - iz_2(k_4 - k_3 - k_2) - ik_1x_1 + ik_4x_2\} \tag{2.2.13}$$

in this way we can compute $\tilde{G}_2$ by evaluating the integral in z_1 and z_2 in (2.2.11). In particular $\int \mathrm{d}^4 z_1$ gives us $(2\pi)^4\delta^{(4)}(k_1 - k_2 - k_3)$, that with $\int \mathrm{d}^4k_3/(2\pi)^4$ sets $k_3 = k_1 - k_2$, then $\int \mathrm{d}^4 z_2$ gives $(2\pi)^4\delta^{(4)}(k_4 - k_1)$ and then $\int \mathrm{d}^4k_4/(2\pi)^4$ sets $k_4 = k_1$. Putting everithing together:

$$\tilde{G}_2(p^2, m^2) = \int \mathrm{d}^4\tilde{x}\, e^{ip\tilde{x}}\, G_2(x_1, x_2) \tag{2.2.14}$$

with $\tilde{x} = x_1 - x_2$ and:

$$\begin{aligned} G_2(x_1, x_2) = {} & D_F(x_1 - x_2) - \\ & - \frac{\lambda^2}{2} \int \frac{\mathrm{d}^4k_1}{(2\pi)^4} \frac{\mathrm{d}^4k_2}{(2\pi)^4} \frac{e^{-ik_1(x_1 - x_2)}}{D(k_1, m)^2\, D(k_2, m)\, D(k_1 - k_2, m)} \end{aligned} \tag{2.2.15}$$

where we introduce:

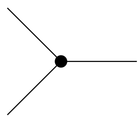
$$D(k, m) = k^2 - m^2 + i\emptyset^+. \tag{2.2.16}$$

Now, the integral over $\tilde{x} = x_1 - x_2$ can be performed giving a $\delta^{(4)}(p - k_1)$ factor, followed by the integral in d^4k_1 fixing $k_1 = p$, so gives us:

$$\begin{aligned} \tilde{G}_2(p^2, m^2) = {} & \frac{i}{D(p, m)} + \\ & + \frac{(-i\lambda)^2}{2} \left(\frac{i}{D(p, m)}\right)^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{i}{D(k, m)} \frac{i}{D(k - p, m)} + \mathcal{O}(\lambda^4). \end{aligned} \tag{2.2.17}$$

We like this result, but we know, from other courses, that Feynman introduced a diagrammatic way to arrive at the same expression avoiding the lenght algebra. For the $\lambda\phi^3$ theory we have the Feynman rules:

- Interaction vertex:



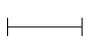
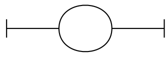
$$= -i\lambda$$

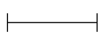
- Internal lines:

$$= \frac{i}{D(k, m)}$$

- Divide by the symmetry factor of the diagram.

In this way we have easly the expression for the correlation function up to $\mathcal{O}(\lambda^2)$:

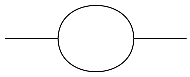
$\tilde{G}_2(p^2, m^2) =$  $+$  $+ \; \mathcal{O}(\lambda^4)$

where we have used the notation  to indicate this is a correlations function and so *propagators are included* on external lines, unlike the case of S-matrix elements where diagrams are amputated.

Note that we must include the symmetry factor of $1/2$ (to the loop) when applying the Feynman rules.

2.3 Identifying the problem - The self energy

The one-loop correction to the propagator, usually reffered to ad *one-loop self energy*, is:

$\Sigma^{(1)}(p^2, m^2) \; = -i$ 

where (using the Feynman rule):

$$\Sigma^{(1)}(p^2, m^2) = \frac{i\lambda}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{D(k, m) D(k-p, m)} \tag{2.3.1}$$

so that (2.2.17) became:

$$\tilde{G}_2(p^2, m^2) = \frac{i}{D(p, m)} \left(1 + \frac{\Sigma^{(1)}(p^2, m^2)}{D(p, m)} + \mathcal{O}(\lambda^4) \right). \tag{2.3.2}$$

Wrote this we can explicit the limit $|k| \to \infty$:

$$\Sigma^{(1)}(p^2, m^2) \quad \longrightarrow \quad \frac{i\lambda}{2} \int_0^\infty \frac{\mathrm{d}|k| \, |k|^3 \, \mathrm{d}^3\Omega}{|k|^4} \propto \int_0^\infty \frac{\mathrm{d}|k|}{|k|} \tag{2.3.3}$$

hence the self-energy has a logaritmic divergence in the UV, and we must include a cut-off to compute the integral:

$$\int_\delta^\Lambda \frac{\mathrm{d}|k|}{|k|} = \log\left(\frac{\Lambda}{\delta}\right). \tag{2.3.4}$$

Understanding how to treat such UV divergences through *renormalizzation*[1], will be the main topic of this course.

[1] Note that including a cut-off is only one way (of many) to dominate the infinity.

2.3.1 The self energy in d dimensions

We can repeat the argument in d-dimension concerning the UV divergence of the one loop self-energy:

$$\Sigma^{(1)[d]}(p^2, m^2) = -i\frac{\lambda^2}{2} I_2^{(1)[d]} \tag{2.3.5}$$

where we write the integral:

$$I_2^{(1)[d]} = \int_k \frac{1}{D(k,m)D(k-p,m)} \tag{2.3.6}$$

with:

$$\int_k = \int \frac{\mathrm{d}^d k}{(2\pi)^4}.$$

In the limit $|k| \to \infty$ we have:

$$I_2^{(1)[d]} \underset{|k|\to\infty}{\longrightarrow} \int \frac{\mathrm{d}|k|\,|k|^{d-1}\mathrm{d}^{d-1}\Omega}{|k|^4(2\pi)^4} \tag{2.3.7}$$

$$= \underline{\int \mathrm{d}|k|\,|k|^{d-5}} \int \frac{\mathrm{d}^{d-1}\Omega}{(2\pi)^d} \tag{2.3.8}$$

therefore the integral underlined for any dimension $d \geq 4$ diverges, and its divergence in $d = 4$ is logarithmic.

Since there is non-zero mass m there is a natural lower band for the energy $|k|$ integral, so using a (upper) cut-off regularisation scale Λ we can get:

$$\Sigma^{(1)[4]}(p^2, m^2) \underset{|k|\to\infty}{\longrightarrow} c \log\left(\frac{\Lambda^2}{m^2}\right) \tag{2.3.9}$$

with c a constant and we use Λ as a UV regulator. We can easily imagine that if we are in the massless limit (notice that this is the problem we can find doing the calculation for a soft Bremsstrahlung in QED) we have some problem even in the lower intagration region.

We could ask ourselves what is the meaning of Σ? Taking ϕ^3, so a vertex with 3 fields. The conservation delta we have in each vertex of the theory imposes us that not all the particle that meet in it are on-shell. Indeed, if a physical particle came with momentum p_1 and mass m, and decades in two final particle, which have to have momentum $\vec{p}_2 = -\vec{p}_3$ and mass m. Now, if we impose the mass-shell on the final state we obtain:

$$E_2 = \sqrt{|\vec{p}_2|^2 + m^2} = E_3$$

but this break the energy conservation because in the initial state we have $E_1 = m$ and in the final $2E_2 > m$. So, the final state particle are not on-shell, they can propagate only for a finite time and space, are not physical, cannot be detected and are quantum fluctuations of the field ϕ. In the experiments we can see something only if these un-physical particles recombine, in another vertex, and give an on-shell particle. These intermediate particle are called *virtual particle*.

As we know, we have to integrate over all possible path, so even on those which do not satisfy $p^2 = m^2$, that are the non-physical solution. The non-physical state contribute less to the path integral. We saw that the problematic configuration are those with large momentum k, so when we consider high energy for very non physical particle.

What we are saying is that: we are considering an high energy theory, but in practice we cannot compute all the contribution to some propagator at that energy. Indeeed, if we increase the energy, for example to a propagator of a photon, does not contribute only a virtual e^+e^-, but a virtual $q\bar{q}$ either, and increasing more the energy we have the gravity too; which unfortunately we are not able to quantize. So, what we are doing is to consider just one contribution to the propagator correction, ignoring all the other infinite possibilities, and is reasonable thet the theory breaks down (remember the KLN theorem from the MSc course *Phenomenology of Fundamental Interaction*).

Chapter 3

$\lambda\phi^3$ theory at 1 loop

Now that we've identified the problem we will consider our study of ϕ^3 theory. This time we specify a generale space-time dimension, d, so we have:

$$S = \int \mathrm{d}^d x \, \mathcal{L} \tag{3.0.1}$$

with the same lagrangian as before:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{3!}\phi^3. \tag{3.0.2}$$

The metric is taken to be *mostly minus* hence:

$$g^{\mu\nu} = \mathrm{diag}\{1, -1, \dots, -1\}. \tag{3.0.3}$$

Before finding a resolution to the problem of UV divergences our first task is to classify the possible divergent graph.

In particular we will see how to renormalize UV divergences in ϕ^3 in $4d$, with a mass shift (and redefining $\langle\phi\rangle$), we will introduce methods for loop integrals, like dimensional regularization and Feynman parameters. We will see how renormalized perturbation theory depends on an arbitrary scale μ_R at fixed order.

Note. In this chapter, as the previous one, we will represent the Feynman diagrams for scalar fields with solid lines, and not with dashed.

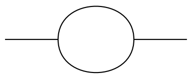
3.1 Superficial degree of divengence

We can perform this analysis for a general *one particle irreducible*, 1PI, graph $\Gamma_E(n, L)$ where:

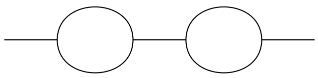
- E the number of external lines

- n the number of internal lines
- L the number of loops.

We didn't defined what are 1PI graphs; we can say that *one particle irreducible* graphs are the ones which cannot be divided in sub-graphs which are 1PI too. For example, the graph:



is 1PI, while the graph:



is not. The UV power counting gives us the *superficial degree of divergence*[1], $\omega(\Gamma_E)$:

$$\omega(\Gamma_E) = dL - 2n \tag{3.1.1}$$

which can be seen as the power of momentum in the integral (we have dL power from $\mathrm{d}^d k$ and a k^{-2} from the internal propagator). The extension to higer loop assumes scaling all internal momenta to ∞ at the same rate. So, if $\omega(\Gamma_E)$ tells us the power of the momentum integrated, we may try to conclude:

- $\omega(\Gamma_E) < 0$ means *convergence.*
- $\omega(\Gamma_E) = 0$ means *logarithmic divergence.*
- $\omega(\Gamma_E) > 0$ means *power low divergence.*

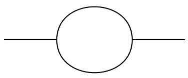
This statement can be made concrete by considering also the superficial degree of divergence of each subgraph:

Weinberg's Theorem: A graph Γ is convergent in the UV if $\omega(\Gamma) < 0$ and $\omega(\Gamma_{sub}) < 0$ for every subgraph Γ_{sub} of Γ.

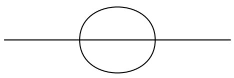
NB. The superficial degree of divergence does not account of potential cancellations between graphs with the same number of external legs and loop order, so contributing to the same correlation function G_n.

[1] We call it *superficial* because the actual degree of divergence can e complicated in gauge theory and some pathological cases. You can see Peskin Let no one ignorant of geometry enter my door.Peskin section §10.1 for detail.

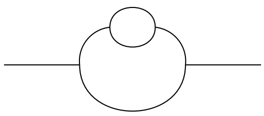
Examples of divergences:



$$\begin{cases} n = 2 \\ E = 2 \\ L = 1 \end{cases} \quad ; \quad \omega(\Gamma_2) = d - 4$$



$$\begin{cases} n = 3 \\ E = 2 \\ L = 2 \end{cases} \quad ; \quad \omega(\Gamma_2) = 2d - 6 = 2(d - 3)$$



$$\begin{cases} n = 5 \\ E = 2 \\ L = 2 \end{cases} \quad ; \quad \omega(\Gamma_2) = 2d - 10 = 2(d - 5)$$

in this last case we might be tempted to say convergent in $d = 4$, but subgraph diverges in $d = 4$ (the one-loop in the upper line), so the Weinberg's theorem is not satisfied.

We may express ω in terms of the dimensions of the fields and couplings. For a good general scalar theory with p-point interaction, so:

$$\mathcal{L}_{int} = -\frac{\lambda}{p!}\phi^p \tag{3.1.2}$$

we find that the number of loops are:

$$L = n - V + 1 \qquad , \qquad n = \frac{pV - E}{2} \tag{3.1.3}$$

so the number of internal lines (n) minus the number of vertices V (because we have in each vertex one conservation delta) minus one global conservation delta. So we can get:

$$\omega(\Gamma) = dL - 2n \tag{3.1.4}$$
$$= d(n - V + 1) - 2n \tag{3.1.5}$$
$$= (d - 1)\frac{(pV - E)}{2} - dV + d \tag{3.1.6}$$
$$= d - E\left(\frac{d - 2}{2}\right) - V\left(d - p\left(\frac{d - 2}{2}\right)\right) \tag{3.1.7}$$
$$= d - E\frac{[\phi]}{[m]} - V\frac{[\lambda]}{[m]} \tag{3.1.8}$$

where we have used, from the lagrangian:

$$[m^2\phi^2] = 2[m] + 2[\phi] \equiv d[m] \tag{3.1.9}$$

$$\implies \quad [\phi] = \frac{(d-2)}{2}[m] \tag{3.1.10}$$

$$[\lambda\phi^p] \equiv d[m] \tag{3.1.11}$$

$$\implies \quad [\lambda] + p[\phi] = d[m] \tag{3.1.12}$$

$$\frac{[\lambda]}{[m]} = d - \left(\frac{d-2}{2}\right)p. \tag{3.1.13}$$

Our ability to renormalize will rely on the number of UV divergences that we may encounter in the correlation functions. We would like a finite number of divergent (sub)-graph. In terms of the dimension of λ we classify cases as:

- **Super-renormalizable**: if $[\lambda]/[m] > 0$ and a graph Γ has $\omega(\Gamma) < 0$, at loop order L, then the additional loop correction to Γ (some values for $d, E, [\phi]/[m]$, higher L mean higher V) will also be convergent (will decrease $\omega(\Gamma)$), so this means a finite number of divergent graphs.

- **Renormalizable**: if $[\lambda]/[m] = 0$ then we have ∞ number of divergent graphs, but only for limited number of extenal lines. However, the number of divergent sub-graphs is finite.

- **Non-renormalizable**: if $[\lambda]/[m] < 0$. Every 1PI graph is divergent if we go to a sufficiently high loop order; as we can see in the relation (3.1.8), where at fixed dimension and external legs we can get $\omega(\Gamma) > 0$ simply increasing the number of vertices, so the number of loops.

We can notice that the expression (3.1.7), with $d = 4$, gives us:

$$\omega(\Gamma) = 4 - E - V(4-p) \tag{3.1.14}$$

which tells us that the superficial degree of divergence is the dimension of the operator that is present inside the loop-integral. So, if we want some $\omega(\Gamma)$ we can consider, together the dimension of spece-time, just the dimension of the different operator that figure inside the integral (fermion, boson, vector). We will use this fact for QED (5.1.28) and QCD (6.1.83).

3.2 Computing $\Sigma^{(1)}$ in dimensional regularization

Having estabilshed, in the section §2.3.1, that the self-energy is divergent we may calculate it only after introducing a *regularization* method.

We've saw that loop integral contains infinities; the regularisation methods try to quantify the information contained in the diagrams and continuing doing physics keeping these infinities.

We already demonstrated a simple *cut-off* would render the integral finite, but unfortunately it breaks Poincaré invariance, so is not predered. We can perform *dimensional regularization* where d is analytically continued to be non-integer:

$$d = 4 - 2\epsilon \qquad , \qquad \epsilon \ll 0 \tag{3.2.1}$$

note that instead 4 we can use d_0 as the dimension for $\mathcal{L}$. Dimensional regularization is an analytic continuation to a continuous dimension, and it preserves many symmetries (Poincaré, abelian and non-abelian gauge) and so it can be generalizes well to QED and QCD. For this reason we may as well introduce it immediately. Later on in the chapter we will list other possibilities.

What we are doing in dimensional regularisation is to change the dimension from 4 to a general non integer d, using the parameter ϵ, ad it's convenient because some integral are easier to compute and the results, with its divergences, will be in term of ϵ. The dimensional regularisation has been introduced because we've saw that our problem came from the dimensionality of the space-time.

Now, we can now simply compute:

$$\Sigma^{(1)[d]} = -i\frac{\lambda^2}{2} I_2^{(1)[d]} \tag{3.2.2}$$

where:

$$I_2^{(1)[d]} = \int_k \frac{1}{D(k,m)D(k-p,m)}. \tag{3.2.3}$$

3.2.1 Feynman and Schwinger parameters

Feynman, and similarly Schwinger[2], notice that it was convenient to observe (you can try it in Mathematica) the relation:

$$\frac{1}{AB} = \int_0^1 \mathrm{d}\alpha_1 \int_0^1 \mathrm{d}\alpha_2 \, \frac{\delta(1-\alpha_1-\alpha_2)}{(\alpha_1 A + \alpha_2 B)^2}. \tag{3.2.4}$$

We will study the generalization to more denominators with arbitrary powers later, but for now consider just 2 denominator. We write, computing the delta for convenience (so we get $\alpha \equiv \alpha_2 = 1 - \alpha_1$):

$$I_2^{(1)[d]} = \int_k \int_0^1 \mathrm{d}\alpha \, \frac{1}{(\alpha D(k-p,m) + (1-\alpha)D(k,m))^2} \tag{3.2.5}$$

[2]Note that some author, such as Smirnov Let no one ignorant of geometry enter my door.Smirnov, call the Schwinger parameters *alpha parameters*.

completing the square in the denominatos yields:

$$\alpha((k-p)^2 - m^2 + i\emptyset^+) + (1-\alpha)(k^2 - m^2 + i\emptyset^+) = \tag{3.2.6}$$
$$= k^2 - m^2 + i\emptyset^+ - 2\alpha k\cdot p + \alpha p^2 \tag{3.2.7}$$
$$= (k-\alpha p)^2 - \alpha p^2 + \alpha^2 p^2 - m^2 + i\emptyset^+ \tag{3.2.8}$$
$$= (k')^2 - \Delta \tag{3.2.9}$$

where we define:

$$\Delta = -\alpha(1-\alpha)p^2 + m^2 - i\emptyset^+ \tag{3.2.10}$$
$$k' = k - \alpha p. \tag{3.2.11}$$

We can now re-order integration in α and k (changing variable from k' to k) to write:

$$I_2^{(1)[d]} = \int_0^1 \mathrm{d}\alpha \int_k \frac{1}{(k^2-\Delta)^2}. \tag{3.2.12}$$

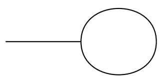
Note. If the numerator has dependence on k we would pick up α dependence in the numerator due to the shift $k' = k - \alpha p$.

3.2.2 The tadpole integral

In $I_2^{(1)[d]}$ (3.2.12) we have a standard integral that we can solve; for this reason we can consider a general **tadpole integral** defined as:

$$T_n^{[d]}(\Delta) = \int_k \frac{1}{(k^2-\Delta)^n}. \tag{3.2.13}$$

A tadpole graph would look like:



The first step to solve (3.2.13) is to Wick rotate into Euclidean kinematics:

$$k^0 = ik_E^0 \qquad , \qquad \vec{k} = \vec{k}_E \tag{3.2.14}$$

such that:

$$k^2 = (k^0)^2 - |\vec{k}|^2 \tag{3.2.15}$$
$$= -(k_E^0)^2 - |\vec{k}_E|^2 \tag{3.2.16}$$
$$= -k_E^2 \tag{3.2.17}$$

and:

$$\mathrm{d}^d k = i\mathrm{d}^d k_E. \tag{3.2.18}$$

The tranformation (remember that the Wick rotation rotate the axis in anti-clock sense) does not cross the poles from the denominator[3], see fig. 3.1, and note that we get the denominator:

$$k^2 - \Delta = (k^0)^2 - |\vec{k}|^2 - \Delta = 0 \tag{3.2.22}$$

$$\implies \quad k^0 = \pm\sqrt{|\vec{k}|^2 + \Delta} = \pm\sqrt{|\vec{k}| + \mathrm{Re}\{\Delta\}} \mp i\emptyset^+ = k^0_\pm . \tag{3.2.23}$$

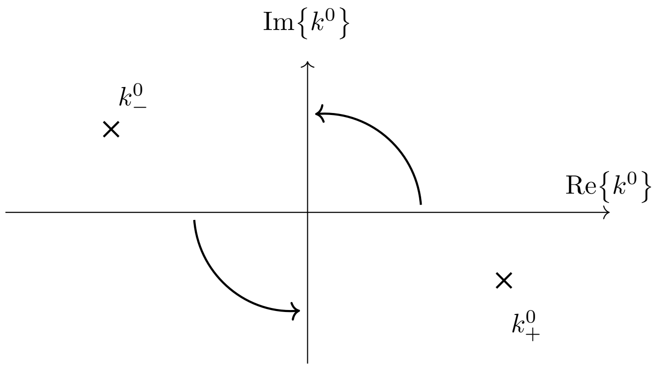


Figure 3.1: Wick rotation.

[3]Note that this is more than a replacement. See Peskin Let no one ignorant of geometry enter my door.Peskin and Sterman Let no one ignorant of geometry enter my door.Sterman. When we perform the Wick rotation we must do it in anti-clock direction. This ensures that the contour transitions from the real axis to the imaginary axis through the 1st and 3rd quadrants, avoiding the poles at $k^0 = \pm\omega \mp i\epsilon$. See fig. 3.1. By Cauchy's theorem, since no poles are enclosed by the closed contour formed by the real axis, the imaginary axis, and the connecting arcs at infinity (which vanish), we have:

$$\oint_C \mathrm{d}k^0 = 0 \qquad , \quad \text{with } k^0 \in \mathbb{C} \tag{3.2.19}$$

but we can write the closed path as:

$$\oint_C \mathrm{d}k^0 = \int_{-\infty}^{+\infty} \mathrm{d}k^0 + \int_{C_{arc}} \mathrm{d}k^0 + \int_{+i\infty}^{-i\infty} \mathrm{d}k^0 = 0 \tag{3.2.20}$$

so the sum of the integral on the real axis, the infinite arc C_{arc} (which goes to zero) and the immaginary axis (followed backwards). Since the integral on C_{arc} vanishes, we get the Wick rotation as:

$$\int_{-\infty}^{+\infty} \mathrm{d}k^0 = \int_{-i\infty}^{+i\infty} \mathrm{d}k^0 \tag{3.2.21}$$

which is the relation (3.2.18) if we set $k^0 = ik^0_E$ (where k^0_E is real and goes from $-\infty$ to $+\infty$).

Therefore:

$$T_n^{[d]}(\Delta) = i(-1)^n \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \frac{1}{(k_E^2 + \Delta)^n} \tag{3.2.24}$$

$$= i(-1)^n \int \frac{|k_E|^{d-1}\, \mathrm{d}|k_E|}{(|k_E|^2 + \Delta)^n} \int \frac{\mathrm{d}^{d-1}\Omega}{(2\pi)^d} \tag{3.2.25}$$

$$= i(-1)^n \int \frac{|k_E|^{d-1}\, \mathrm{d}|k_E|}{(|k_E|^2 + \Delta)^n} \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{3.2.26}$$

where we used the well-known angular integral:

$$\int \frac{\mathrm{d}^{d-1}\Omega}{(2\pi)^d} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

where we have the *Gamma function*:

$$\Gamma(x) = \int_0^x \mathrm{d}z\, z^{x-1} e^{-z}.$$

We can see that the integration in $|k_E|$ in (3.2.26) is also straightforward, because we can call $|k_E| = x$, then do the sostitution $x^2 = t$ and then $y = \Delta/(t + \Delta)$ to get:

$$\int_0^\infty \mathrm{d}x \frac{x^{d-1}}{(x^2 + \Delta)^n} \underset{x^2=t}{=} \int_0^\infty \frac{\mathrm{d}t}{2} \frac{t^{d/2-1}}{2(t + \Delta)^n} \tag{3.2.27}$$

$$= \frac{1}{2} \int_0^1 \mathrm{d}y\, \Delta^{d/2-n}\, y^{n-1-d/2}\, (1 - y)^{d/2-1} \tag{3.2.28}$$

we can reconize the Euler beta function, defined as:

$$B(a, b) = \int_0^1 \mathrm{d}x\, x^{a-1} (1 - x)^{b-1} \tag{3.2.29}$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} \tag{3.2.30}$$

so, with $a = n - d/2$ and $b = d/2$, we have:

$$\int \frac{|k_E|^{d-1}\, \mathrm{d}|k_E|}{(|k_E|^2 + \Delta)^n} = \frac{\Delta^{d/2-n}}{2} \frac{\Gamma\left(n - d/2\right) \Gamma\left(d/2\right)}{\Gamma(n)} \tag{3.2.31}$$

so, putting this relation inside (3.2.26), we have a result for the tadpole:

$$T_n[d](\Delta) = \frac{i(-1)^n}{(4\pi)^{d/2}} \Delta^{d/2-n} \frac{\Gamma\left(n - d/2\right)}{\Gamma(n)}. \tag{3.2.32}$$

3.2.3 The self-energy in the massless case

Before completing $I_2^{(1)[d]}(p^2, m^2)$ we can try the simpler case, at one loop in d dimension, with $m = 0$. We have to use the result (3.2.32) in d dimension with $n = 2$, so noticing that (3.2.10) became:

$$\Delta = -\alpha(1-\alpha)p^2 \tag{3.2.33}$$

dropping $i\emptyset^+$, we have:

$$T_2^{[d]}(-\alpha(1-\alpha)p^2) = \frac{i}{(4\pi)^{d/2}} \left(-\alpha(1-\alpha)p^2\right)^{d/2-2} \frac{\Gamma\left(2-d/2\right)}{\Gamma(2)} \tag{3.2.34}$$

so the integral in the self energy is:

$$\begin{aligned} I_2^{(1)[d]}(p^2, 0) &= \int_0^1 \mathrm{d}\alpha\, T_2^{[d]}(\Delta) &(3.2.35)\\ &= \frac{i}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}\alpha \frac{\Gamma(2-d/2)}{\Gamma(2)} \left(-\alpha(1-\alpha)p^2\right)^{d/2-2} &(3.2.36)\\ &= \frac{i}{(4\pi)^{d/2}} (-p^2)^{d/2-2}\, \Gamma\left(2-\frac{d}{2}\right) B\left(\frac{d}{2}-1, \frac{d}{2}-1\right) &(3.2.37)\\ &= \frac{i}{(4\pi)^{d/2}} (-p^2)^{d/2-2} \frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma^2\left(\frac{d}{2}-1\right)}{\Gamma\left(d-2\right)} &(3.2.38) \end{aligned}$$

where we have used the relation between the Beta function and the Gamma (3.2.30). In $d = 4 - 2\epsilon$ dimensions, we have:

$$\frac{d}{2} - n \longrightarrow \frac{4-2\epsilon}{2} - 2 = -\epsilon$$

so, computing (3.2.38) for $d = 4 - 2\epsilon$ we have:

$$I_2^{(1)[4-2\epsilon]}(p^2, 0) = \frac{i}{(4\pi)^{2-\epsilon}} (-p^2)^{-\epsilon} \frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} \tag{3.2.39}$$

where the UV pole can be made explicit through use of the property:

$$\Gamma(1+z) = z\Gamma(z)$$

in particular:

$$\Gamma(\epsilon) = \frac{\Gamma(1+\epsilon)}{\epsilon} \quad , \quad \Gamma(2-2\epsilon) = (1-2\epsilon)\Gamma(1-2\epsilon) \tag{3.2.40}$$

so we can conclude:

$$I_2^{(1)[4-2\epsilon]}(p^2, 0) = \frac{iC_\Gamma}{(4\pi)^2} \frac{1}{(1-2\epsilon)\epsilon} (-p^2)^{-\epsilon} \tag{3.2.41}$$

$$\text{with the coefficient:} \qquad C_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{(4\pi)^{-\epsilon}\Gamma(1-2\epsilon)}. \tag{3.2.42}$$

This is not the end of the story because we may also expand in ϵ to observe the analytic structure:

$$I_2^{(1)[4-2\epsilon]}(p^2,0) = \frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon} + 2 - \gamma_E + \log(4\pi) - \log\left(-p^2\right) + \mathcal{O}(\epsilon)\right) \tag{3.2.43}$$

where we expand $(-p^2)^{-\epsilon}$, and $(4\pi)^{\epsilon}$, with the log:

$$\left(\frac{-p^2}{4\pi}\right)^{-\epsilon} = \exp\left\{-\epsilon\log\left(\frac{-p^2}{4\pi}\right)\right\} \tag{3.2.44}$$

$$\sim 1 - \epsilon\Big[\log\left(-p^2\right) - \log\left(4\pi\right)\Big] \tag{3.2.45}$$

and we have the Euler-Mascheroni's constant:

$$\gamma_E = 0.57721 \tag{3.2.46}$$

that comes from the expansion of Gamma function:

$$\Gamma(\epsilon) \sim \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon) \tag{3.2.47}$$

in particular we have to use:

$$\begin{aligned}
&\Gamma(1+\epsilon) = \epsilon\Gamma(\epsilon) \sim \epsilon\left(\frac{1}{\epsilon} - \gamma_E\right) = 1 - \epsilon\gamma_E \\
&\Gamma(1-\epsilon) = -\epsilon\Gamma(-\epsilon) \sim -\epsilon\left(-\frac{1}{\epsilon} - \gamma_E\right) = 1 + \epsilon\gamma_E \\
&\qquad\Longrightarrow\quad \Gamma^2(1-\epsilon) \sim (1+\epsilon\gamma_E)^2 \sim 1 + 2\epsilon\gamma_E \\
&\Gamma(1-2\epsilon) = (-2\epsilon)\Gamma(-2\epsilon) \sim (-2\epsilon)\left(-\frac{1}{2\epsilon} - \gamma_E\right) = 1 + 2\epsilon\gamma_E.
\end{aligned}$$

3.2.4 The self-energy with mass dependence

Now focus on the same problem, with $n = 2$ in d dimension, but with $m \neq 0$; we have (dropping again $\emptyset^+$ in $\hat{\Delta}$):

$$I_2^{(2)[d]}(p^2,m^2) = \int_0^1 \mathrm{d}\alpha\, T_2^{[d]}\Big(-\alpha(1-\alpha)p^2 + m^2\Big) \tag{3.2.48}$$

$$= \frac{i}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\int_0^1 \mathrm{d}\alpha\, \left(-\alpha(1-\alpha)p^2 + m^2\right)^{d/2-2} \tag{3.2.49}$$

putting $d = 4 - 2\epsilon$:

$$I_2^{(2)[4-2\epsilon]}(p^2,m^2) = \frac{i}{(4\pi)^{2-\epsilon}}\Gamma(\epsilon)(m^2)^{-\epsilon}\int_0^1 \mathrm{d}\alpha\left(\underbrace{1-\alpha(1-\alpha)\frac{p^2}{m^2}}_{\equiv\hat{\Delta}}\right)^{-\epsilon} \tag{3.2.50}$$

we can reconize the same UV divergence:

$$\Gamma(\epsilon) = \frac{1}{\epsilon}\Gamma(1+\epsilon) \tag{3.2.51}$$

and the integral in α is finite so we can expand it in ϵ (using the same method of (3.2.44)), and then integrate in α:

$$\int_0^1 \mathrm{d}\alpha\, \hat{\Delta}^{-\epsilon} = \int_0^1 \mathrm{d}\alpha \left(1 - \epsilon \log(\hat{\Delta}) + \dots\right) \tag{3.2.52}$$

so:

$$I_2^{(1)[4-2\epsilon]}(p^2, m^2) = \frac{i}{(4\pi)^{2-\epsilon}}\Gamma(1+\epsilon)(m^2)^{-\epsilon}\left(\frac{1}{\epsilon} - \int_0^1 \mathrm{d}\alpha\, \log(\hat{\Delta}) + \mathcal{O}(\epsilon)\right) \tag{3.2.53}$$

deriving a closed form for the integral over α is a straightforward exercise (you can see the linked file).

Exercise 1.

1. From the definitions:

$$\beta_\pm = \frac{1}{2}(1 \pm \beta) \quad , \quad \beta = \sqrt{1 - \frac{4m^2}{p^2}} \tag{3.2.54}$$

   show that $\beta_+\beta_- = m^2/p^2$.

2. Use the transformation $\alpha = y + \beta_+$ to show:

$$\hat{\Delta} = \frac{y(y+\beta)}{\beta_+\beta_-}.$$

3. Complete the integration over y to show:

$$\int_0^1 \mathrm{d}\alpha \log\left(\hat{\Delta}\right) = -2 + \beta \log\left(-\frac{\beta_+}{\beta_-}\right).$$

The final result (thanks to the exercise) is therefore:

$$I_2^{(1)[4-2\epsilon]}(p^2, m^2) = \frac{i\Gamma(1+\epsilon)m^{-2\epsilon}}{(4\pi)^{2-\epsilon}}\left(\frac{1}{\epsilon} + 2 - \beta\log\left(-\frac{\beta_+}{\beta_-}\right)\right) + \mathcal{O}(\epsilon) \tag{3.2.55}$$

$$= \frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon} + 2 + \log(4\pi) - \gamma_E - \log(m^2) - \right.$$
$$\left. - \beta\log\left(-\frac{\beta_+}{\beta_-}\right)\right) + \mathcal{O}(\epsilon). \tag{3.2.56}$$

Putting this into the definition of $\Sigma^{(1)}$ (3.2.2) we find (at the first order):

$$\Sigma^{(1)[4-2\epsilon]}(p^2, m^2) = \frac{\lambda^2}{2(4\pi)^2}\left(\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0)\right). \tag{3.2.57}$$

The result is valid for every value of p (which is contained inside the $\beta_\pm$), even if the expression:

$$\hat{\Delta} = 1 - \alpha(1-\alpha)\frac{p^2}{m^2} \tag{3.2.58}$$

gives us problem in the region $p^2 \to \infty$, because it compared inside a square root, making it immaginary. We don't have problem because we can take the result (3.2.57) and analitically continue it in the region problematic.

We can also see that the logarithm in the result is:

$$\log\left(\frac{\sqrt{1-\frac{4m^2}{k^2}}-1}{\sqrt{1-\frac{4m^2}{k^2}}+1}\right) \tag{3.2.59}$$

and we can have a cut, due to the square root, for $k^2 \geq 4m^2$, but this is not surprising and this has a physical meaning very interesting. Indeed, if we remember the Lehmann-Källén representation (LK) for the propagator:

$$\Delta_F = \frac{Z}{k^2 - m_R^2 + i\epsilon} + \int_{4m^2}^{+\infty} d\mu^2 \frac{\rho(\mu^2)}{k^2 - \mu^2 + i\epsilon}$$

where the cut begins exactly at $4m^2$. We also recall that $4m^2$ was the threshold for creating (in the center of mass) two real, on-shell particles at rest. We will coming back to this value for the cut when we will talk about the Cutkowski rules in the last chapter. What we have is:

- For $k^2 < 4m^2$, our particle is stable (it cannot decay), the self-energy is real, and we are in the region of virtual "vacuum fluctuations" which are reabsorbed almost instantaneously.

- In the region $k^2 > 4m^2$, the particle can easily decay into two particles, and this is equivalent, per LK, to no longer having a single isolated particle state, but rather a continuum of many-particle states.

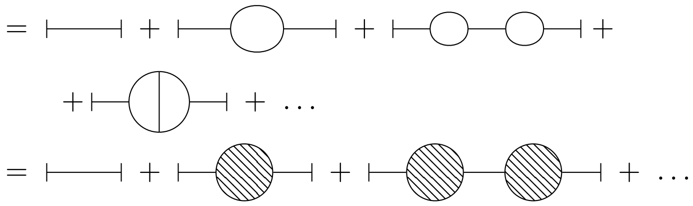
So we have a threshold which gives us a different behavior if we are above or below it. Passing the threshold we goes from virtual correction to real process. Remember, when we have immaginary parts we are in presence of real particle production.

In particular, from the optical theorem, we have that the imaginary part of the self-energy is linked to the cross section (or width of decay). Knowing also that $\rho(\mu^2)$ gives us the probability to find states with invariant mass μ. The analytical manifestation of Σ's cut (from log) is due to the fact that, physically, at that energy a new channel of creation of new particles opens.

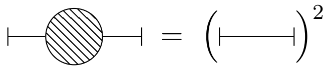
3.3 Interpretation of the divergence: renormalized 2-point function

Let's return to the perturbative expansion of $\tilde{G}_2$:

$\tilde{G}_2(p^2, m^2) =$



where we have the 1PI contribution:



$\Sigma(p^2, m^2)$

with:

$$\Sigma(p^2, m^2) = \sum_{L=1}^{\infty} \Sigma^{(L)}(p^2, m^2) \tag{3.3.1}$$

which is the self-energy containing all not facturisable diagrams (1PI). This way we obtain:

$$\tilde{G}_2(p^2, m^2) = \frac{i}{D(p.m)} \left(1 + \frac{\Sigma}{D(p,m)} + \frac{\Sigma^2}{D(p,m)^2} + \dots \right) \tag{3.3.2}$$

$$= \frac{i}{D(p,m)} \frac{1}{1 - \frac{\Sigma(p^2,m^2)}{D(p,m)}} \tag{3.3.3}$$

where we are re-summing to all order (Dyson):

$$\tilde{G}_2(p^2, m^2) = \frac{i}{p^2 - m^2 - \Sigma(p^2, m^2) + i\emptyset^+}. \tag{3.3.4}$$

So the self-energy, containing the UV divergence, can be interpreted as a *mass shift* to the tree-level propagator. So, in the end $\Sigma(p^2, m^2)$ shifts the propagator's pole.

It is therefore possible to absorb the divergence into a re-defined mass parameter m_R, which is the only physical quantity (that we measure in the experiments), to fix finite terms:

$$m_R^2 = m^2 - \delta m^2(\kappa) \tag{3.3.5}$$

where κ is the (generic) *scheme dependence* which cancel the divergence in Σ and m is the bare mass. But with the result (3.2.57) we are able to compute $\delta m^2(\kappa)$ up to one-loop. In facts, writing:

$$\delta m^2(\kappa) = \sum_{L=1}^{\infty} \delta m^{2\,(L)}(\kappa) \tag{3.3.6}$$

and:

$$m^2 + \Sigma(p^2, m^2) = m_R^2 + \delta m^{2\,(1)}(\kappa) + \Sigma^{(1)}(p^2, m_\kappa^2) + \mathcal{O}(\lambda^4) \tag{3.3.7}$$

where we impose the cancellation of divergences:

$$\delta m^{2\,(1)}(\kappa) + \Sigma^{(1)}(p^2, m^2) = \text{UV finite} \tag{3.3.8}$$

$$\Longrightarrow \qquad \delta m^{2\,(1)}(\kappa) = \frac{\lambda^2}{2(4\pi)^2}\left(-\frac{1}{\epsilon} + \kappa\right) \tag{3.3.9}$$

and in order to make a physical prediction we must fix κ, so we must define a *renormalization scheme*.

So, just to clarify, we have some indefinite integral, we rewrite them using the *regularisation*, which allows us to rewrite the integral keeping the divergences and see the finite parts, we define the *counterterms* to absorb these divergences, but doing that we have an entire family of parametrization. We fix the parameter in the counterterms with the *renormalization scheme*, which fix a value for κ.

We must also address the mass dimension of our loop corrections. There is a new dimension hidden inside the coupling in dimensional regularisation. Recall (3.1.13):

$$\frac{[\lambda]}{[m]} = \left(d - 3\left(\frac{d-2}{2}\right)\right) \tag{3.3.10}$$

$$\underset{d=4-2\epsilon}{=} (1+\epsilon) \tag{3.3.11}$$

so we can make the scale explicit by using:

$$\lambda_R \mu_R^\epsilon = \lambda \tag{3.3.12}$$

where:

$$[\lambda_R] = [m] \qquad , \qquad [\mu_R] = [m] \tag{3.3.13}$$

so we have λ_R with the same dimension of *bare* coupling in $d = 4$. The finite, *renormalized*, 2-point correlation function (3.3.4) is therefore (in $4d$ taking $\epsilon \to 0$):

$$\tilde{G}_{2,R} = -i\Gamma_{2,R}^{-1} \tag{3.3.14}$$

where we have:

$$\begin{aligned}\Gamma_{2,R}(p^2, m_r^2; \kappa, \mu_R^2, \lambda_R^2) = p^2 - m_R^2 + i\emptyset^+ + \\ + \frac{\lambda_R^2}{2(4\pi)^2}\Big(2 + \log(4\pi) - \gamma_E - \kappa \\ + \log\left(\frac{\mu_R^2}{m_R^2}\right) - \beta\log\left(-\frac{\beta_+}{\beta_-}\right)\Big) + \\ + \mathcal{O}(\lambda_R^4)\end{aligned} \tag{3.3.15}$$

where we can still choose a specific value, so a renormalization scheme, for κ.

3.3.1 Renormalization of the lagrangian and counter-terms

We may apply the change of parameters already at the level of the lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda}{3!}\phi^3 \tag{3.3.16}$$

where we have now specified m_0 and λ_0 ad *bare* quantities. After the transformation:

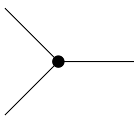
$$m_0^2 = m_R^2 + \delta m^2 \tag{3.3.17}$$
$$\lambda_0 = \lambda_R \mu_R^\epsilon \tag{3.3.18}$$

we obtain:

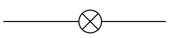
$$\mathcal{L} = \mathcal{L}\Big|_{\substack{m_0\to m_R\\ \lambda_0\to\lambda_R\mu^\epsilon}} - \frac{1}{2}\delta m^2\phi^2 \tag{3.3.19}$$

where, computing with Feynman rules, the first term gives us the same contribute as before:

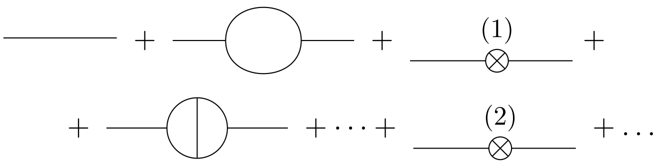
 $= \dfrac{i}{D(p, m_R)}$

 $= -i\lambda_R\mu_R^\epsilon$

but the second term gives us:

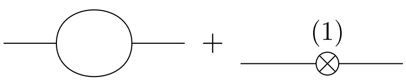
 $= -i\delta m^2$

which is the *UV counter-term from local operator*. Hence we generate renormalized (finite) quantities order by order:

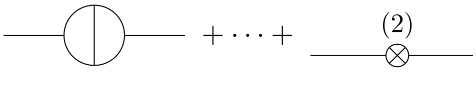


where we identify the order:

 $= \mathcal{O}(\lambda_R^0)$ (3.3.20)



$= \mathcal{O}(\lambda_R^2)$ (3.3.21)



$= \mathcal{O}(\lambda_R^4).$ (3.3.22)

3.3.2 Renormalization schemes

We see that our finite $\tilde{G}_{2,R}$ still has an ambiguity that must be fixed before the procedure is predictive. Indeed, we saw that the parameter m_0 is not the physical mass but we must specify how m_R is connected to a physical quantity, which will fix the scheme, so κ and μ_R.

We define the physical mass m_p to be at the pole of the 2-point function:

$$\tilde{G}_{2,p} = \frac{i}{p^2 - m_p^2 + i\emptyset^+} \tag{3.3.23}$$

and now consider options for the scheme.

1. Minimal subtraction (MS) Only poles from divergent graphs retained, so we fix $\kappa_{MS} = 0$. We can then state that:

$$-iG_{2R}^{-1} = \Gamma_{2,\text{phys}} \tag{3.3.24}$$

with:

$$\lim_{p^2 \to m_{\text{phys}}^2} \Gamma_{2,\text{phys}} = i\emptyset^+ \tag{3.3.25}$$

which we match to the renormalized function:

$$\Gamma_{2,MS}(p^2, m_R^2; \mu_R^2, \lambda_R^2) \tag{3.3.26}$$

as follows:

$$\lim_{p^2 \to m_{\text{phys}}^2} \Gamma_{2,MS}(p^2, m_R^2; \mu_R^2, \lambda_R^2) = i\emptyset^+ \tag{3.3.27}$$

where, from before we have (3.3.15), so:

$$m_{\text{phys}}^2 - m_R^2 + \frac{\lambda_R^2}{2(4\pi)^2}\left(2 + \log(4\pi) - \gamma_E + \log\left(\frac{\mu_R^2}{m_R^2}\right) - \right.$$
$$\left. - \beta_{\text{phys}} \log\left(-\frac{\beta_{\text{phys}+}}{\beta_{\text{phys}-}}\right)\right) = \mathcal{O}(\lambda_R^4) \tag{3.3.28}$$

where we have to use the expressions (3.2.54):

$$\beta_{\text{phys}} = \sqrt{1 - \frac{4m_R^2}{m_{\text{phys}}^2}} \quad , \quad \beta_{\text{phys}\pm} = \frac{1}{2}\left(1 \pm \beta_{\text{phys}}\right). \tag{3.3.29}$$

So we have fixed the scheme to a measurement at $p^2 = m_{\text{phys}}^2$ and found a (relatively complicated) relation between $m_{R,MS}^2$ and m_p^2. At leading order:

$$m_{R,MS}^2 = m_{\text{phys}}^2 + \mathcal{O}(\lambda_R^2) \tag{3.3.30}$$

so we simplify the relation above by replacing m_R with m_{phys} inside the terms at $\mathcal{O}(\lambda_R^2)$:

$$m_{R,MS}^2 = m_{\text{phys}}^2 + \frac{\lambda_R^2}{2(4\pi)^2}\Big(2 + \log(4\pi) - \gamma_E + \log\left(\frac{\mu_R^2}{m_{\text{phys}}^2}\right) - \\ - \frac{\pi}{\sqrt{3}}\Big) + \mathcal{O}(\lambda_R^4) \tag{3.3.31}$$

where, as we can see, we have[4]:

$$\beta \log\left(-\frac{\beta_+}{\beta_-}\right)\Bigg|_{\substack{p^2=m_{\text{phys}}^2 \\ m_R^2=m_{\text{phys}}^2}} = \frac{\pi}{\sqrt{3}}. \tag{3.3.32}$$

So, this scheme, although it's the simplest, gives us a physical mass a lot different from the renormalized one.

2. Modified minimal subtraction ($\overline{\textbf{MS}}$) We can also include some additional constants in the choice of the κ which may help to include an additional finite constant in $\delta m^{2\,(1)}$:

$$\kappa_{\overline{MS}} = \log(4\pi) - \gamma_E \tag{3.3.33}$$

which can also be seen as a change in normalisation subtracting:

$$\frac{(4\pi)^\epsilon e^{-\epsilon\gamma_E}}{\epsilon} \tag{3.3.34}$$

rather than $1/\epsilon$. This relation between $m_{R,\overline{MS}}^2$ and m_{phys}^2 is particularly useful for higher order correction, ad it is:

$$m_{R,\overline{MS}} = m_{\text{phys}}^2 + \frac{\lambda_R^2}{2(4\pi)^2}\left(2 + \log\left(\frac{\mu_R^2}{m_{\text{phys}}^2}\right) - \frac{\pi}{\sqrt{3}}\right) \tag{3.3.35}$$

instead (3.3.31).

3. Fixed momentum subtraction (MOM) These are scheme where we fix the momentum p^2, in this way the variable became the difference between the momentum and the scale choosen.

[4]Because we can see that $\beta_{phys} = i\sqrt{3}$ and we have:

$$-\frac{\beta_{phys+}}{\beta_{phys-}} = -\frac{1+i\sqrt{3}}{1-i\sqrt{3}} = (e^{i\pi})\frac{2e^{i\pi/3}}{2e^{-i\pi/3}} = e^{-i\pi/3}$$

which has to be put inside the (natural) logarithm.

In this case we fix the 2-point function at a value[5] $p^2 = -M^2$ and impose:

$$\Gamma_{2,MOM}(-M^2, m_R^2; \kappa_{MOM}, \mu_R^2, \lambda_R^2) = -M^2 - m_R^2 + i\emptyset^+ \tag{3.3.36}$$

from which we determine:

$$\kappa_{MOM} = 2 + \log(4\pi) - \gamma_E + \log\left(\frac{\mu_R^2}{m_R^2}\right) - \beta_M \log\left(-\frac{\beta_{M+}}{\beta_{M-}}\right) \tag{3.3.37}$$

and so:

$$\begin{aligned}\Gamma_{2,MOM}(p^2, m_R^2; -M^2, \lambda_R^2) &= p^2 - m_R^2 + i\emptyset^+ + \\ &+ \frac{\lambda_R^2}{2(4\pi)^2}\left(-\beta_R \log\left(-\frac{\beta_{R+}}{\beta_{R-}}\right) + \beta_M \log\left(-\frac{\beta_{M+}}{\beta_{M-}}\right)\right) + \mathcal{O}(\lambda_R^4).\end{aligned} \tag{3.3.38}$$

The relation to the physical mass can be determined as before:

$$\lim_{p^2 \to m_{\text{phys}}} \Gamma_{2,MOM}(p^2, m_R^2; M^2, \lambda_R) = i\emptyset^+ \tag{3.3.39}$$

so we have:

$$\begin{aligned}m_{R,MOM}^2 = m_{\text{phys}}^2 + \frac{\lambda_R^2}{2(4\pi)^2}\Bigg(&- \frac{\pi}{\sqrt{3}} + \\ &+ \beta(-M^2, m_{\text{phys}}) \log\left(-\frac{\beta_+(-M^2, m_{\text{phys}})}{\beta_-(-M^2, m_{\text{phys}})}\right)\Bigg) + \mathcal{O}(\lambda_R^4).\end{aligned} \tag{3.3.40}$$

So, it's diassapeared the scale μ_R, but we gain the scale $-M^2$; indeed, these kind of scheme allow us to change the scale.

4. On-shell subtraction In this scheme we impose $m_R^2 = m_{\text{phys}}^2$:

$$\lim_{p^2 \to m_R} \Gamma_{2,R}(p^2, m_R^2; \kappa_{OS}, \mu_R^2, \lambda_R^2) = i\emptyset^+ \tag{3.3.41}$$

so we have:

$$\kappa_{OS} = 2 + \log(4\pi) - \gamma_E + \log\left(\frac{\mu_R^2}{m_R^2}\right) - \frac{\pi}{\sqrt{3}}. \tag{3.3.42}$$

NB. For theories like QCD where m_{phys} is not in the perturbative regime we cannote use the on-shell scheme. Indeed if we put the quark in on-shell scheme we are in the confinement regime, so in the non perturbative region.

[5] The minus sign is important, because doing so we don't worry about the branch-cut in the logaritm and there are not immaginary parts.

3.4 Complete renormalization of ϕ^3 in 4d

Having completed our analysis of the two point function we need to consider all other potentially divergent graph. The result so far for the lagrangian in terms of renormalized mass and coupling is (remember the results (3.3.9) and (3.3.19)):

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_R^2\phi^2 - \frac{\lambda_R\mu_R^\epsilon}{3!}\phi^3 + \frac{1}{2}\delta m^2\phi^2 \tag{3.4.1}$$

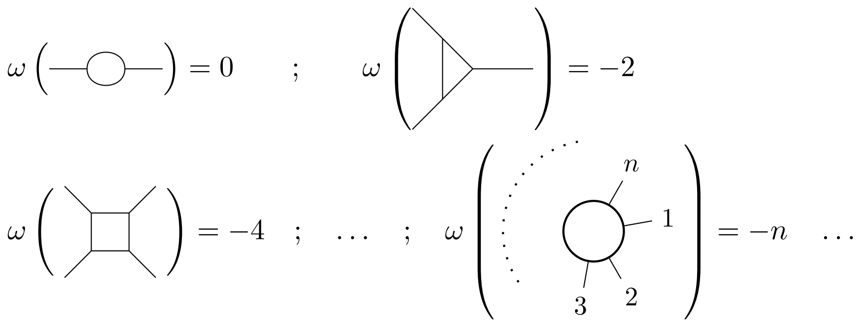
where:

$$\delta m^2 = \frac{\lambda_R^2}{2(4\pi)^2}\left(\frac{1}{\epsilon}+\kappa\right) + \mathcal{O}(\lambda_R^4) \tag{3.4.2}$$

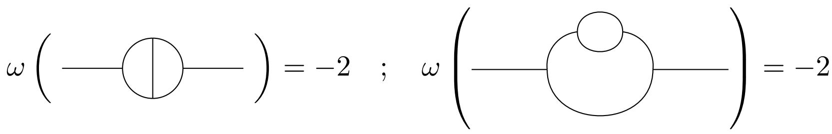
which cancel the divergence in $\tilde{G}_2$, so that $\tilde{G}_{2,R}$ is finite. In $d=4$ we know we have $[\lambda_R] = [m]$, so we only have a finite number of divergent graph. Remember:

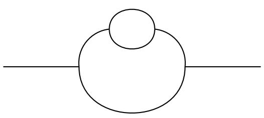
$$\omega(\Gamma) = 4 - E - V \tag{3.4.3}$$

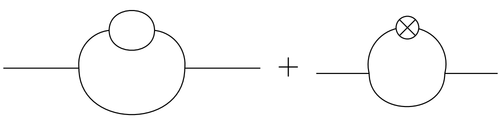
we have, at **1-loop**:



so no further 1-loop divergences. At **2-loop**:

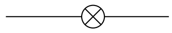


but  has one-loop divergent subgraph which is cancelled exactly by the counter-term insertion into the one-loop graph:



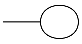
= finite (3.4.4)

therefore we can determine that $\delta m^{2\,(2)} = 0$, so:

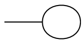
(2)
 $= 0$ however.

This pattern continues, to all-loop orders, so the one-loop counter-term is sufficient to render all correlation function finite; so we find δm^2 is complete at one-loop.

There is one subtlety, which is:

$$\omega\left(\right.$$  $$\left.\right) = 4 - 1 - 1 = 2 \tag{3.4.5}$$

so we have that the tadpole graph is also divergent; indeed:



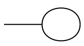
$$\begin{aligned} &= -\frac{i\lambda_R}{2}\mu_R^\epsilon \int_k \frac{i}{D(k, m_R)} &(3.4.6)\\ &= \frac{\lambda_R}{2}\mu_R^\epsilon I_1^{(1)[4-2\epsilon]} &(3.4.7)\\ &= -i\frac{\lambda_R}{2}\frac{\Gamma(-1+\epsilon)}{\Gamma(1)}\frac{1}{(4\pi)^{2-\epsilon}}(m_R^2)^{1-\epsilon}\mu_R^\epsilon &(3.4.8)\\ &= \frac{i\lambda_R\mu_R^{-\epsilon}m_R^2}{2(4\pi)^2}\frac{\Gamma(1+\epsilon)}{(1-\epsilon)\epsilon}\left(\frac{\mu_R^2}{m_R^2}\right)^\epsilon &(3.4.9)\\ &= i\frac{\lambda_R}{2(4\pi)^2}m_R^2\left(\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0)\right) &(3.4.10) \end{aligned}$$

where we applied the relation $\Gamma(z+1) = z\Gamma(z)$ and its expansion (3.2.47). This graph contributes to the vacuum expectation value:

$$\langle 0|\,\phi\,|0\rangle_{\text{free}} = 0 \tag{3.4.11}$$

and we know for an interacting theory:

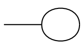
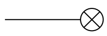
$$\langle 0|\,\phi\,|0\rangle_{\text{int}} = \langle 0|\,\phi\, S_{\text{int}}(\phi)\,|0\rangle_{\text{free}} + \dots \tag{3.4.12}$$

$=$  $+ \dots$ (3.4.13)

Due to the fact that (3.4.10) is non zero we have to add to the theory a term that absorb its divergence. Indeed, the vacuum energy represents a physical observable and therefore we may renormalize the divergence into the vacuum. We would like to:

$$\langle 0|\,\phi\,|0\rangle_R = \langle\phi\rangle_{\text{phys}} \tag{3.4.14}$$

$$= 0 \tag{3.4.15}$$

$=$  $+$  $= 0$ (3.4.16) where we have taken $——\otimes$ the tadpole counter-term to cancel completly the divergence. The counter-term is introduced as a linear term into the lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_R^2\phi^2 - \frac{\lambda_R\mu_R^\epsilon}{6}\phi^3 - \frac{1}{2}\delta m^2\phi^2 + \delta\langle\phi\rangle\phi \tag{3.4.17}$$

where:

$$\delta\langle\phi\rangle \;=\; ——\otimes \;=\; -\,—\bigcirc \tag{3.4.18}$$

in order to cancel the quantum corrections exactly.

Usually this process, to cancel the tadpole's contribute, works so well that we don't consider it and we don't modify the lagrangian; but, in the Standard Model it's not always true.

3.5 Alternative regularisation scheme

Historically, dimensional regularization, analized starting from the section §3.2, was introduced in 1972 (from 't Hooft, Veltman and Giandiagi, Bollini). Other options are listed below.

Cut-off regularisation (e.g. Schwinger, Feynman etc.) In this scheme we impose an upper limit for the momentum; in this scheme Wick rotation and Feynman parametrisation work in the same way and we have:

$$I_2^{(1)} = \int_k^\Lambda \frac{1}{D(k,m)D(k-p,m)} \tag{3.5.1}$$

$$= \int_0^1 \mathrm{d}\alpha \int^\Lambda \frac{\mathrm{d}^4k}{(k^2-\Delta)^2} \tag{3.5.2}$$

$$= i\int_0^1 \mathrm{d}\alpha \int_0^\Lambda \frac{\mathrm{d}|k_E|\,|k_E|^3}{(k_E^2+\Delta)^2}\frac{2\pi^2}{(2\pi)^4} \tag{3.5.3}$$

$$= i\int_0^1 \mathrm{d}\alpha \frac{1}{16\pi^2}\int_0^{\Lambda^2} \mathrm{d}y\,\frac{y}{(y+\Delta)^2} \tag{3.5.4}$$

$$\approx \frac{i}{16\pi^2}\int_0^1 \mathrm{d}\alpha\left(\log\left(\frac{\Lambda^2}{\Delta}\right)-1\right) \tag{3.5.5}$$

where in (3.5.3) we perform the Wick rotation and compute the integral for the angular coordinate.

Lattice regularisation We can apply this scheme discretizing space-time (minimum spacing implies maximum energy), because also this procedure regulates UV divergences. Indeed, if we have a finite spacing we cannot go to infinite energy. You can see the M. Nebbia's notes for detail. We can keep gauge invariance but we break Poincaré invariance. If we write:

$$I_2^{(1)[d]}(p^2,m^2) = \int_k \left(\frac{1}{D(k,m)D(k-p,m)} - \frac{1}{D^2(k,\mu)} + \frac{1}{D^2(k,\mu)} \right) \tag{3.5.6}$$

by subtracting at fixed scale μ, then the divergent integral:

$$\int_k \frac{1}{D^2(k,\mu)} \propto \log\left(\frac{1}{\mu^2\alpha^2}\right) \tag{3.5.7}$$

with $x^\mu = n^\mu a$, where we indicate a like the lattice spacing, and we have $-\pi/a \le k^\Lambda \le \pi/a$.

Pauli-Villars regularisation In this method we introduce new, large mass, states such the each propagator becomes:

$$\frac{1}{D(p,m)} \longrightarrow \frac{1}{D(p,m)} - \frac{1}{D(p,M)} \tag{3.5.8}$$

$$= \frac{m^2 - M^2}{D(p,m)D(p,M)}. \tag{3.5.9}$$

The additional propagator factors enter the Feynman parameters and regulate the integrals but all expressions depend on M. In facts the additional power of momentum downstairs give us faster UV convergence, and we can still proceed with Feynman parameters, so keep integral in 4d, and they will depend on M.

For QED, with this method, we can maintain gauge invariance.

Analytic regularisation (see Bogolinbov, Zinnermann etc.) We start from the observation that:

$$\frac{\partial}{\partial p^2}\Sigma^{(1)[4]}(p^2,m^2) \tag{3.5.10}$$

is finite. From here we can begin to differentiate under the integral. We have the relation:

$$\frac{\partial}{\partial p^2} = \frac{p^\mu}{2p^2}\frac{\partial}{\partial p^\mu} \tag{3.5.11}$$

$$\frac{\partial}{\partial p^\mu}\frac{1}{D(k-p,m)} = \frac{(k-p)_\mu}{D(k-p,m)^2} \tag{3.5.12}$$

$$\frac{\partial}{\partial p^\mu}\Sigma^{(1)[4]}(p^2,m^2) = \frac{-i\lambda^2}{2}\frac{1}{2p^2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{p\cdot(k-p)}{D(k,m)D(k-p,m)^2}. \tag{3.5.13}$$

The additional powers of k downstairs are sufficient to regulate the divergence in the UV. We can perform the Feynman parametrisation via:

$$\frac{1}{A^2B} = 2\int_0^1 \mathrm{d}\alpha\, \alpha \frac{1}{(\alpha A + (1-\alpha)B)^3} \tag{3.5.14}$$

which bring us to:

$$\int_k \frac{p\cdot(k-p)}{D(k,m)D(k-p,m)^2} = 2\int_0^1 \mathrm{d}\alpha\,\alpha \int_k \frac{p\cdot(k-p)}{((k-\alpha p)^2-\Delta)^3} \tag{3.5.15}$$

with:

$$\Delta = -\alpha(1-\alpha)p^2 + m^2 \tag{3.5.16}$$

and so:

$$\int_k \frac{p\cdot(k-p)}{D(k,m)D(k-p,m)^2} = 2\int_0^1 \mathrm{d}\alpha\,\alpha \int_k \frac{p\cdot(k+\alpha p-p)}{(k^2-\Delta)^3}. \tag{3.5.17}$$

Now rewrite (3.5.10), so we get:

$$\frac{\partial}{\partial p^2}\Sigma^{(1)[4]}(p^2,m^2) = \frac{-i\lambda^2}{2}\frac{1}{p^2}\int_0^1 \mathrm{d}\alpha\,\alpha \int_k \frac{k\cdot p-(1-\alpha)p^2}{(k^2-\Delta)^3} \tag{3.5.18}$$

but we have an antisymmetric integrand over a symmetric integration range, so we can state:

$$\int_k \frac{k\cdot p}{(k^2-\Delta)^n} = 0 \tag{3.5.19}$$

and we obtain:

$$\begin{aligned}
\frac{\partial}{\partial p^2}\Sigma^{(1)[4]} &= \frac{i\lambda^2}{2}\int_0^1 \mathrm{d}\alpha\,\alpha(1-\alpha)\,T_3^{[4]} && (3.5.20)\\
&= \frac{i\lambda^2}{2}\int_0^1 \mathrm{d}\alpha\,\alpha(1-\alpha)\left(\frac{-i}{(4\pi)^4}\frac{1}{2\Delta}\right) && (3.5.21)\\
&= -\frac{\lambda^2}{4}\frac{1}{(4\pi)^2}\frac{\partial}{\partial p^2}\int_0^1 \mathrm{d}\alpha\,\log\left(-\alpha(1-\alpha)p^2+m^2\right) && (3.5.22)
\end{aligned}$$

where we revrite $1/\Delta$ as the derivative of a logaritm. In the end we have that the self energy is equal to a finite integral:

$$\Sigma^{(1)[4]} = -\frac{\lambda^2}{4(4\pi)^2}\int_0^1 \mathrm{d}\alpha\,\log(\Delta) + C \tag{3.5.23}$$

where in C, the constant that contains the boundary value, we have the divergence. We can fix the constant using the renormalization scheme condition, e.g. thew *on-shell scheme*:

$$\Sigma^{(1)[4]}(m^2,m^2) = 0 \tag{3.5.24}$$

so we get:

$$\Sigma^{(1)[4]}(p^2,m^2) = -\frac{\lambda^2}{4(4\pi)^2}\left(\int_0^1 \mathrm{d}\alpha\,\log(\Delta) - \int_0^1 \mathrm{d}\alpha\,\log\left(\Delta\Big|_{p^2=m^2}\right)\right) \tag{3.5.25}$$

$$= \frac{-\lambda^2}{4(4\pi)^2}\int_0^1 \mathrm{d}\alpha\,\log\left(\frac{1-\alpha(1-\alpha)\frac{p^2}{m^2}}{1-\alpha(1-\alpha)}\right). \tag{3.5.26}$$

Baikov's method and Mellin-Barnes representation This part is apart from the exam's material, but I prepared some notes about *Mellin-Bernes representation*, the *Integration by Parts method* and the *Baikov's method*, which can be used to calculate some Feynman diagrams; you can see them in my personal page.

3.6 Renormalization of ϕ^3 in 6d at one-loop

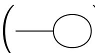
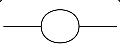
Now we can study how to renormalize in 6 dimension the ϕ^3 theory. We can notice that in 6 dimension the coupling constant, λ, is dimensionless, so for what we said at the end of the section §3.1, the theory is renormalizable, but in a more intricate way than in 4d. We start with the bare lagrangian:

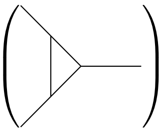
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{3!}\phi_0^3 \tag{3.6.1}$$

in which all quantities are denoted a bare using a subscript zero. The superficial degree of divergence is:

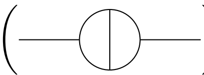
$$\omega\left(\Gamma\right) = 6 - 2E \tag{3.6.2}$$

so:

ω () $= 4$; ω () $= 2$

ω () $= 0$ and < 0 for $E > 0$.

This means we must also take care of divergence in the 3-point vertex function. Since ω is independent from the number of vertices (V) the higher loop corrections to the 1-, 2- and 3- point functions also diverge (not just subgraphs), e.g.:

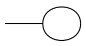
ω () $= 2.$ (3.6.3)

We will first compute the divergent graphs and then show how they may be absorbed into a redefinition of the fields and parameters in the lagrangian.

3.6.1 Divergent graphs at 1-loop

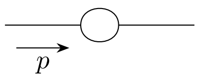
In this section we will compute directly using the bare lagrangian.

Let's start with the **tadpole**:



$$
\begin{aligned}
&= -\frac{i\lambda_0}{2}\, iT_1^{[6-2\epsilon]}(m_0^2) && (3.6.4)\\
&= \frac{\lambda_0}{2}\frac{-i}{(4\pi)^{3-\epsilon}}(m_0^2)^{2-\epsilon}\frac{\Gamma(-2+\epsilon)}{\Gamma(1)} && (3.6.5)\\
&= -\frac{i\lambda_0}{2(4\pi)^3}\frac{(4\pi)^3\Gamma(1+\epsilon)m_0^4(m_0^2)^{-\epsilon}}{\epsilon(1-\epsilon)(2-\epsilon)} && (3.6.6)\\
&= -i\frac{\lambda_0}{2}\frac{1}{2(4\pi)^3}m_0^4\left(\frac{1}{\epsilon}+\frac{3}{2}-\gamma_E+\log{(4\pi)}-\log{(m_0^2)}+\mathcal{O}(\epsilon)\right). && (3.6.7)
\end{aligned}
$$

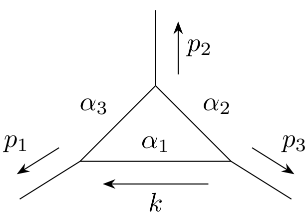
We can analize the **bubble** (using the Wick rotation and the Feynman parametrization):



$$
\begin{aligned}
&= -\frac{(i\lambda_0)^2}{2}\int_0^1 \mathrm{d}\alpha\, i^2 T_2^{[6-2\epsilon]}\left(-\alpha(1-\alpha)p^2+m_0^2\right) && (3.6.8)\\
&= \frac{i\lambda_0^2}{2(4\pi)^3}\left(\frac{1}{\epsilon}\left(-m_0^2+\frac{p^2}{6}\right)+\mathcal{O}(\epsilon^0)\right) && (3.6.9)
\end{aligned}
$$

for the term $\mathcal{O}(\epsilon^0)$ see Mathematica. Now we can see that the divergence has a kinematic dependence, so changing dimension change the kinematic dependence. We will fix this momentum dependence, which can be problematic, later.

For the **triangle** we have:



$$
= (-i\lambda_0)^3\int_k \frac{i^3}{D(k,m)D(k-p_1,m)D(k-p_1-p_2,m)} \tag{3.6.10}
$$

we can use the relation (we have a 3 point function, so we need 3 Feynman parameters):

$$
\frac{1}{ABC} = \int\prod \mathrm{d}\alpha_i \frac{2\delta(1-\alpha_1-\alpha_2-\alpha_3)}{(\alpha_1 A+\alpha_2 B+\alpha_3 C)^3} \tag{3.6.11}
$$

so we have:

$$
= \lambda_0^3\int_0^1 \mathrm{d}\alpha_1\int_0^{1-\alpha_1}\mathrm{d}\alpha_2\int_k \frac{2}{(k^2-\Delta_3)^3} \tag{3.6.12}
$$

where we write:

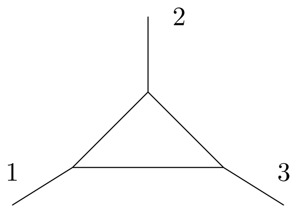
$$
\Delta_3 = m_0^2-(1-\alpha_1)\alpha_1 p_1^2-(1-\alpha_2)\alpha_2 p_2^2-2\alpha_1\alpha_2 p_1\cdot p_2 \tag{3.6.13}
$$

so:

$$= -i\lambda_0^3 2 \int_0^1 \mathrm{d}\alpha_1 \int_0^{1-\alpha_1} \mathrm{d}\alpha_1\, T_3^{[6-2\epsilon]}(\Delta_3) \tag{3.6.14}$$

$$= (-i\lambda_0)(-i\lambda_0^2) \frac{2\Gamma(\epsilon)}{(4\pi)^{3-\epsilon}\Gamma(3)} \int_0^1 \mathrm{d}\alpha_1 \int_0^{1-\alpha_1} \mathrm{d}\alpha_2\, \Delta_3^{-\epsilon} \tag{3.6.15}$$

expanding up to $\mathcal{O}(\epsilon^0)$ is quite difficult in closed form (arise polylogarithms with complicated arguments), but keeping just $1/\epsilon$ the result is quite simple:

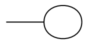
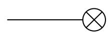


$$= \frac{(-i\lambda_0)\lambda_0^2}{2(4\pi)^3} \left(\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0) \right) \tag{3.6.16}$$

and again see Mathematica for the other orders.

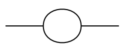
3.6.2 Counter-terms and the renormalized lagrangian

Even in 6 dimensions the *tadpole divergence* is absorbed into the vacuum expectation value, as in the $d = 4$ case, imposed by introducing a linear counter-term matching the tadpole graph:

 $+$ 

$$= 0 \tag{3.6.17}$$

where, as we saw before, the counter term introduces a linear term in $\mathcal{L}$, that we denote as $\delta\langle\phi\rangle$. The *bubble divergence* has changed it's functional form and is no longer absorbed by a simple mass shift (because now we have a kinematic dependence). We saw in (3.6.9):



$$\sim \frac{1}{\epsilon} \left(-m_0^2 + \frac{p^2}{6} \right) \tag{3.6.18}$$

where the first term can be obtained via a redefinition of the mass and a mass counter-term $\sim \delta m^2 \phi^2$, while the second term, dependent on the momentum, can be obtained from a redefinition[6] of the field ϕ and a counter-term $\sim \partial_\mu \phi \partial^\mu \phi \delta\phi$. These re-definition are called *mass* and *wave-function*

[6] Note that this kind of divergence, so as p^2, can be reabsorbed in a lagrangian term because is quadratic and we can write it as a double derivative; otherwise, if we have different power of p, we could have some problem. For example, if we have something like p^4 we could reabsorb it with a 4-derivative terms, at the same time if we have function of the momentum, like logarithm, exponential or something else, we can write the analogous term in the lagrangian but expanded in series.

renormalization respectively. This can be written clearly introducing *renormalization constant* Z_m and Z_ϕ:

$$m_0^2 = Z_m m_R^2 \tag{3.6.19}$$

$$\phi_0 = \sqrt{Z_\phi}\phi_R \tag{3.6.20}$$

where:

$$\delta_x = Z_x - 1 \qquad , \quad \text{for } x = m, \phi. \tag{3.6.21}$$

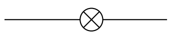
Applying the transformations (3.6.19) and (3.6.20) to the lagrangian leads to:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_R\partial^\mu\phi_R - \frac{1}{2}m_R^2\phi_R^2 + \mathcal{L}_{int} + \underbrace{(Z_\phi - 1)}_{\delta\phi=\delta_2}\frac{1}{2}\partial_\mu\phi_R\partial^\mu\phi_R - \underbrace{(1 - Z_m Z_\phi)}_{\delta_3}\frac{1}{2}m_R^2\phi_R^2 \tag{3.6.22}$$

from which we obtain, renaming:

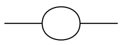
$$\delta_\phi = \delta_2 \qquad ; \qquad \delta_3 = (Z_m Z_\phi - 1)$$

a counter-term Feynman rule:

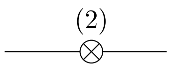


$$= i\left(\delta_2 p^2 - \delta_3 m_R^2\right). \tag{3.6.23}$$

The renormalization constant can then be fixed at 1-loop via:



$+$



$$= \text{UV finite} \tag{3.6.24}$$

for all values of p^2 and m_R^2.

We still have to deal with $\mathcal{L}_{int}$ and the renormalization of the coupling and absorbing the vertex divergence.

The vertex divergence The divergence in the triangle function requires another counter-term that can be obtained through a redefinition of the coupling:

$$\lambda_0 = Z_\lambda \lambda_R \mu_R^\epsilon \qquad , \qquad \frac{[\lambda_R]}{[m]} = 0 \tag{3.6.25}$$

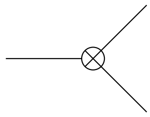
where μ_R is necessary to keep λ_R dimensionless. So we have:

$$\mathcal{L}_{int} = \frac{1}{3!}\lambda_0\phi_0^3 \tag{3.6.26}$$

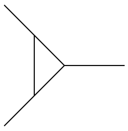
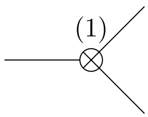
$$= \frac{\lambda_R\mu_R^\epsilon}{3!}\underbrace{Z_\lambda Z_\phi^{3/2}}_{\equiv Z_1 = 1+\delta_1}\phi_R^3 \tag{3.6.27}$$

$$= \frac{\lambda_R\mu_R^\epsilon}{3!}(1+\delta_1)\phi_R^3. \tag{3.6.28}$$

This is *charge renormalization* and leads to a new counter-term with Feynman rule:


$$= -i\frac{\lambda_R \mu^\epsilon}{6}\delta_1 \tag{3.6.29}$$

which can be fixed (at one-loop) using:

 + 

$$= \text{UV finite.} \tag{3.6.30}$$

In the mass' counterterm case is easy to see its meaning, because we saw the Dyson's porpagator resummed, we notice that the physical mass is not the bare mass and we need to include something to it ot make it physical. In the vertex case it's not so easy to see the function of the divergent graph. Now, we can see the divergence as a quantum fluctuation at high energy, that happen in a very small region in space. So, if we see the loop (3.6.16) from long distance we can see it as a point (3.6.29), so as a correction for the coupling constant.

NB. We have $\delta_3 = \delta_m + \delta_\phi + \delta_m\delta_\phi$, using $\delta_\phi = \delta_2$, but the product term is always of higher order in perturbation theory than the linear terms so:

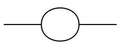
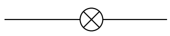
$$\delta_m^{(1)} = \delta_3^{(1)} - \delta_\phi^{(1)} \tag{3.6.31}$$
$$\delta_m^{(2)} = \delta_3^{(2)} - \delta_\phi^{(2)} - \delta_3^{(1)}\delta_\phi^{(1)} \tag{3.6.32}$$

we also have that:

$$Z_1 = Z_\lambda Z_\phi^{3/2} \quad \Longrightarrow \quad \delta_\lambda^{(1)} = \delta_1^{(1)} - \frac{3}{2}\delta_\phi^{(1)}. \tag{3.6.33}$$

3.6.3 ($\overline{\text{MS}}$) renormalization constants at 1-loop

We know have all the ingradients we need to complete the computation of δ_ϕ, δ_m and δ_λ. Explicitly:

 + 
$$= \text{UV finite.} \tag{3.6.34}$$

that implies:

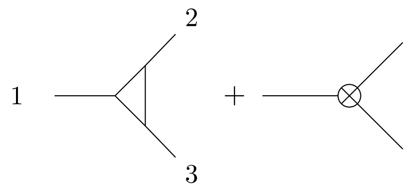
$$\frac{i\lambda_R^2\mu_R^{2\epsilon}}{2(4\pi)^3}\underbrace{\frac{(4\pi)^\epsilon\Gamma(1+\epsilon)}{\epsilon}}_{=C_\Gamma/\epsilon}\left(-m_R^2 + \frac{p^2}{6}\right) + i\left(\delta_2^{(1)}p^2 - \delta_3^{(1)}m_R^2\right) = 0 \tag{3.6.35}$$

so we have:

$$\delta_2^{(1)} = -\frac{\lambda_R^2 \mu_R^{2\epsilon}}{2(4\pi)^3} C_\Gamma \frac{1}{6\epsilon} \tag{3.6.36}$$

$$\delta_3^{(1)} = -\frac{\lambda_R^2 \mu_R^{2\epsilon}}{2(4\pi)^3} C_\Gamma \frac{1}{\epsilon}. \tag{3.6.37}$$

Also we have the relation:



$$= \text{UV finite} \tag{3.6.38}$$

which implies:

$$(-i\lambda_R \mu_R^\epsilon) \frac{\lambda_R^2 \mu_R^{2\epsilon}}{2(4\pi)^3} C_\Gamma \left(\frac{1}{\epsilon}\right) + (-i\lambda_R \mu_R^\epsilon) \delta_1^{(1)} = 0 \tag{3.6.39}$$

which gives us:

$$\delta_1^{(1)} = -\frac{\lambda_R^2 \mu_R^{2\epsilon}}{2(4\pi)^3} C_\Gamma \left(\frac{1}{\epsilon}\right). \tag{3.6.40}$$

Back in terms of $\delta_\phi^{(1)}, \delta_m^{(1)}$ and $\delta_\lambda^{(1)}$ we have (with $N_\lambda = \lambda_R^2 \mu_R^{2\epsilon} C_\Gamma / 2(4\pi)^3$):

$$N_\lambda^{-1} \delta_\phi^{(1)} = -\frac{1}{6\epsilon} \tag{3.6.41}$$

$$N_\lambda^{-1} \delta_m^{(1)} = -\frac{1}{\epsilon} + \frac{1}{6\epsilon} = -\frac{5}{6\epsilon} \tag{3.6.42}$$

$$N_\lambda^{-1} \delta_\lambda^{(1)} = -\frac{1}{\epsilon} - \frac{3}{2}\left(-\frac{1}{6\epsilon}\right) = -\frac{3}{4\epsilon}. \tag{3.6.43}$$

3.6.4 Other renormalization schemes

For on-shell and momentum subtraction (MOM) we impose a stricter contidion that determined the finite terms. Since there are now 3 renormalization constants we must find an extendeded set of condition. These are:

$$\lim_{p^2 \to x} \Gamma_{2,R}(p^2, m_R^2; k_2, k_3, \mu_R^2, \lambda_R^2) = i\emptyset^+ \tag{3.6.44}$$

$$\lim_{p^2 \to x} \frac{\partial \Gamma_{2,R}}{\partial p^2} = 1 \tag{3.6.45}$$

with $x = -M^2$ for MOM and $x = m_{phys}^2 = m_R^2$ for the on-shell scheme.

In MOM and $\overline{MS}$ schemes we must also derive the relation between m_R^2 and m_{phys}^2. For k, we impose (with the minus sign for convetion):

$$\lim_{p_i \to \lambda_{\text{phys}} \substack{\text{value where} \\ \text{measured}}} G_{3,R}(p_1, p_2; k_1, \mu_R^2, m_R^2, \lambda_R^2) = -\lambda_{\text{phys}} \tag{3.6.46}$$

for the on-shell scheme and (for example):

$$\lim_{p_i \to p_{sym}} G_{3,R}(p_1, p_2; k_1, \mu_R^2, m_R^2, \lambda_R^2) = -\lambda_R \tag{3.6.47}$$

for the MOM scheme (no quantum corrections to the vertex at $p_i = p_{sym}$). The symmetric point is defined as:

$$p_i^2 = -\frac{2}{3}M^2 \quad \Longrightarrow \quad p_i \cdot p_j = \frac{M^2}{3} \quad \text{for } i \neq j \tag{3.6.48}$$

e.g.:

$$p_3^2 = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2. \tag{3.6.49}$$

3.6.5 Renormalization scale dependence

In general we saw that the physical quantities has to be connected with the renormalized quantities, and this connection happen via the renormalization scheme.

Having fixed the 3 scheme constant k_i with the conditions on Γ_2, $\frac{\partial}{\partial p}\Gamma_2$ and G_3, we have determined m_R and λ_R in terms of same fixed values (physical measurements) m_{phys} and λ_{phys} but the dependence on μ_R^2 remains. We are in the situation with:

$$m_R = m_R(m_{phys}, \lambda_{phys}, \mu_R, \kappa) \tag{3.6.50}$$

$$\lambda_R = \lambda_R(m_{phys}, \lambda_{phys}, \mu_R, \kappa). \tag{3.6.51}$$

This is a feature of the re-organized renormalized perturbative series. Eventually it must be that the dependences goes away as we sum to all orders - the lagrangian after all has no changed:

$$\mathcal{L}(\phi_0, m_0, \lambda_0) = \mathcal{L}(\phi_R, m_R, \lambda_R; \mu_R) \tag{3.6.52}$$

which means the bare correlation functions do not depend on μ_R:

$$\tilde{G}_0(p_1, \ldots, p_n; \lambda_0) = FT\left(\langle 0| T[\phi_0(x_1)\phi_0(x_2)\ldots] |0\rangle\right) \tag{3.6.53}$$

$$= Z_\phi^{n/2}(\mu_R, \lambda_R(\mu_R))\, FT(\langle 0| T[\phi_R(x_1)\phi_R(x_2)\ldots] |0\rangle) \tag{3.6.54}$$

$$= Z_\phi^{n/2}(\mu_R, \lambda_R(\mu_R))\, \tilde{G}_R(p_1, \ldots, p_n; \lambda_R(\mu_R), \mu_R) \tag{3.6.55}$$

where with FT we indicate the Fourier transform. Indeed, what we do in practice is to measure some process, in particular we want the S_F matrix, that we can compare with the theoretical prediction $S(m_R, \lambda_R, \mu_R, \kappa)$ and it gives us the relations (3.6.50) and (3.6.51). But, to obtain the S matrix we have to compute the Green's functions, which mustn't depends on the normalisation of the fields, so we can write the relation (3.6.55).

In M. Nebbis'a notes we can see the calculation to say that the S matrix elements does not change with the renormalization:

$$S(m_R, \lambda_R, \mu_R, \kappa; p_i, p_j) = S(m_0, \lambda_0; p_i, p_j) \tag{3.6.56}$$

which is a remarkable result because it allows us to make the same prediction with the renormalized parameters, instead the bare ones (which is not physical detectable).

However, in (3.6.55) the μ_R dependence of $\tilde{G}_R$ must cancel exactly with the μ_R dependence of $Z_\phi^{n/2}$. In other words we have:

$$0 = \frac{\partial}{\partial \mu_R}\tilde{G}_0 = \frac{\partial}{\partial \mu_R}\left(Z_\phi^{n/2}\right)\tilde{G}_R + Z_\phi^{n/2}\frac{\partial}{\partial \mu_R}\tilde{G}_R. \tag{3.6.57}$$

We will explore the conseguence of this scale dependence when studying the *renormalization group* (see section §7.3).

What we are saying is that we could just as well have defined the same theory at a different scale μ_R. By "the same theory", we mean a theory whose bare Green's functions, are given by the same functions of the bare coupling constant λ_0 and the renormalization one λ_R. We end founding that the renormalized Green's functions are numerically equal to the bare Green's functions, up to a rescaling by powers of the field strength renormalization Z

Chapter 4

Loop integration methods in dimensional regularization

In this chapter will expand the techniques used in chapter §3 and develop methods need to continue our discussion to gauge theories.

We will study Feynman and Schwinger parameter representations of one-loop integrals in dimensional regularisation. We will found an all-loop parametrisation using F and u Symanzik polynomials. Also, we will study the tensor reduction in parameter and momentum space, to conclude defining che Clifford Algebra for $d = 4 - 2\epsilon$.

Further reading:

- *Scattering amplitudes in QFT* [chapter 4], Badger, Henn, Plefka, Zoia (Springer 2024), `https://arxiv.org/pdf/2306.05976`.

- *Feynman Integrals* [Comprehensive: 800+ pages], Weinzierl (Springer 2022), `https://arxiv.org/pdf/2201.03593`.

Note. In this chapter, as the previous one, we will represent the Feynman diagrams for scalar fields with solid lines, and not with dashed.

4.1 Feynman and Schwinger parameters

We came across 3 examples of *Feynman parametrisation* and we saw the representations:

$$\frac{1}{AB} = \int_0^1 \mathrm{d}\alpha \, \frac{1}{[A\alpha + B(1-\alpha)]^2} \tag{4.1.1}$$

$$\frac{1}{AB^2} = 2\int_0^1 \mathrm{d}\alpha \, \alpha \frac{1}{[A\alpha + B(1-\alpha)]^2} \tag{4.1.2}$$

$$\frac{1}{ABC} = 2\int_0^1 \mathrm{d}\alpha_1 \int_0^{1-\alpha_1} \mathrm{d}\alpha_2 \frac{1}{[A\alpha_1 + B\alpha_2 + C(1-\alpha_1-\alpha_2)]^3} \tag{4.1.3}$$

but we can write the general version:

$$\frac{1}{\prod_{i=1}^n A_i^{\nu_i}} = \frac{\Gamma\left(\sum_{i=1}^n \nu_i\right)}{\prod_{i=1}^n \Gamma\left(\nu_i\right)} \int_0^1 \prod_{i=0}^n \mathrm{d}\alpha_i \, \alpha_i^{\nu_i - 1} \frac{\delta(1-\sum_i \alpha_i)}{\left(\sum_i D_i \alpha_i\right)^{\sum_i \nu_i}} \tag{4.1.4}$$

which is valid also for non-integer ν_i. A similar transformation is provided through *Schwinger parameters*:

$$\frac{1}{D^\nu} = \frac{(-1)^n}{\Gamma(\nu)} \int_0^\infty \mathrm{d}t \, t^{\nu-1} \exp\{-tD\}. \tag{4.1.5}$$

The 1-loop bubble example, so (3.2.3), becomes (performing at some point the shift $k \to k + t_2 p/(t_1+t_2)$):

$$I_2^{(1)[d]} = \int_k \int_0^\infty \mathrm{d}t_1 \int_0^\infty \mathrm{d}t_2 \, \exp\{-t_1 D(k,m) - t_2 D(k-p,m)\} \tag{4.1.6}$$

$$= \int_k \int_{t_i} \exp\left\{-(t_1+t_2)(k^2 - m^2) - \frac{t_1 t_2}{t_1+t_2} p^2\right\} \tag{4.1.7}$$

$$= \int_k \int_{t_1} \exp\left\{(ak^2 + b)\right\} \tag{4.1.8}$$

where we write:

$$a = t_1 + t_2 \qquad ; \qquad b = m^2(t_1+t_2) - \frac{t_1 t_2}{t_1+t_2} p^2. \tag{4.1.9}$$

The integral over k can be performed after a Wick rotation:

$$I_2^{(1)[d]} = i \int_{t_i} \int_{k_E} \exp\left\{-a k_E^2\right\} \exp\{b\} \tag{4.1.10}$$

$$= \frac{i}{(4\pi)^{d/2}} \int_{t_i} \exp\{b\} a^{-d/2} \tag{4.1.11}$$

where we can see the exponential doesn't depend on the loop momentum.

4.1.1 Tensor integrals with Feynman/Schwinger parameters

This section is very important in gauge theories where we will find loop momentum dependent numerators, which require further analysis. We did find one case already, where we analized $\frac{\partial}{\partial p^2}\Sigma$ for ϕ^3, although it was handeled in an easy manner. Let's try the example again using Schwinger parameters; we define:

$$I_2^{(1)[d]\mu}(p,m) = \int_k \frac{k^\mu}{D(k,m)D(k-p,m)} \tag{4.1.12}$$

now if we perform the loop momentum shift:

$$k \longrightarrow k + t_2 p/(t_1+t_2) \tag{4.1.13}$$

that we used to put the denominator in standard form, it also affects the numerator so:

$$I_2^{(1)[d]\mu} = \int_k \int_{t_i} \left(k + \frac{t_2 p}{t_1+t_2}\right)^\mu \exp\{ak^2 - b\} \tag{4.1.14}$$

$$= p^\mu \int_{t_i} \frac{t_2}{t_1+t_2} \frac{i}{(4\pi)^{d/2}} a^{-d/2} \exp\{-b\} \tag{4.1.15}$$

where we performed the rotation, we integrated over k_E and have used (for symmetry reasons):

$$\int_k k^\mu \exp\{ak^2\} = 0. \tag{4.1.16}$$

The parametric integration seems more difficult, but we can use the symmetry[1] in $t_1 \leftrightarrow t_2$ to write:

$$I_2^{(1)[d]\mu} = \frac{p^\mu}{2} \int_{t_i} \left(\frac{t_2}{t_1+t_2} + \frac{t_1}{t_1+t_2}\right) \frac{i}{(4\pi)^{d/2}} a^{-d/2} \exp\{-b\} \tag{4.1.17}$$

$$= \frac{p^\mu}{2} I_2^{(1)[d]} \tag{4.1.18}$$

where we reconized the result (4.1.11).

We define the **tensor rank of a Feynman integral** as the total power of all loop momenta in the numerator of the integrand. As we increase the

[1]If I associate t_1 to the other propagator, t_2 to the remaining and reperform the computation, we obtain the same prefactor but with $t_1 \leftrightarrow t_2$.

tensor rank we encouter new loop momentum integrals; some examples are:

$$\int_k \exp\{ak^2\} = \frac{i}{(4\pi)^{d/2}} a^{-d/2} = \kappa \tag{4.1.19}$$

$$\int_k k^\mu \exp\{ak^2\} = 0 \tag{4.1.20}$$

$$\int_k k^\mu k^\nu \exp\{ak^2\} = -\frac{1}{2a} g^{\mu\nu} \kappa \tag{4.1.21}$$

$$\int_k k^{\mu_1} \dots k^{\mu_{2n+1}} \exp\{ak^2\} = 0 \qquad n \in \mathbb{Z} \tag{4.1.22}$$

$$\int_k k^{\mu_1} \dots k^{\mu_{2n}} \exp\{ak^2\} = -\frac{1}{2a} \sum_{i=2}^{2n} g^{\mu_1 \mu_i} \int_k \frac{k^{\mu_1} \dots k^{\mu_{2n}}}{k^{\mu_1} k^{\mu_i}} \exp\{ak^2\} \tag{4.1.23}$$

with $n \in \mathbb{Z}$ in all the case. In (4.1.21) the metric tensor is the only tensor we can use, over performed the integration, to give tensor structure (I_2, after the shift, depends on p^2). We have the last case (4.1.23) as an exercise; though it's simple to prove (4.1.21) as follow: firstly we need to identify a basis of tensor structures (after integration), there are no external momenta so there's only one tensor structure $g^{\mu\nu}$ possible:

$$\int_k k^\mu k^\nu \exp\{ak^2\} = A g^{\mu\nu} \tag{4.1.24}$$

where A is a *scalar form factor* (to be determined). Contracting on both sides with $g_{\mu\nu}$ and using $g^{\mu\nu} g_{\mu\nu} = g^\mu_\mu = d$ gives:

$$dA = \int_k k^2 \exp\{ak^2\} \tag{4.1.25}$$

$$= \int_k \frac{\partial}{\partial a} \exp\{ak^2\} \tag{4.1.26}$$

$$= \frac{\partial}{\partial a} \int_k \exp\{ak^2\} \tag{4.1.27}$$

$$= \frac{\partial}{\partial a} \left(\frac{i}{(4\pi)^{d/4}} a^{-d/2} \right) \tag{4.1.28}$$

$$= -\frac{d}{2} \frac{1}{a} \frac{i}{(4\pi)^{d/2}} a^{-d/2} \tag{4.1.29}$$

$$dA = -\frac{d}{2a} \kappa \tag{4.1.30}$$

$$A = -\frac{1}{2a} \kappa \tag{4.1.31}$$

as previously stated.

Similar expressions are obtained using Feynman parameters after following the same steps. For example:

$$I_2^{(1)[d]\mu} = \int_0^1 \mathrm{d}\alpha \int_k \frac{k^\mu + \alpha p^\mu}{(k^2-\Delta)^2} \tag{4.1.32}$$

$$= p^\mu \int_0^1 \mathrm{d}\alpha\, \alpha T_2^{[d]}(\Delta) \tag{4.1.33}$$

and:

$$I_2^{(1)[d]\mu\nu} = \int_0^1 \mathrm{d}\alpha \int_k \frac{(k^\mu + \alpha p^\mu)(k^\nu + \alpha p^\nu)}{(k^2-\Delta)^2}. \tag{4.1.34}$$

Exercise 2. Show that we may write the rank 2 tensor bubble integral as:

$$I_2^{(1)[d]\mu\nu} = p^\mu p^\nu \int_0^1 \mathrm{d}\alpha\, \alpha^2 T_2^{[d]}(\Delta) + g^{\mu\nu}\frac{1}{d}\int_0^1 \mathrm{d}\alpha \left(T_1^{[d]}(\Delta) + \Delta T_2^{[d]}(\Delta)\right) \tag{4.1.35}$$

where:

$$\Delta = -\alpha(1-\alpha)p^2 + m^2. \tag{4.1.36}$$

You can also may use the result:

$$\int_k \frac{k^\mu k^\nu}{(k^2-\Delta)^n} = \frac{1}{d} g^{\mu\nu} \int \frac{1}{(k^2-\Delta)^n}. \tag{4.1.37}$$

Exercise 3. Write down the general tensor decomposition for a rank-4 tensor tadpole:

$$T_n^{[d]\mu_1\cdots\mu_4}(\Delta) = \int_k \frac{k^{\mu_1} k^{\mu_2} k^{\mu_3} k^{\mu_4}}{(k^2-\Delta)^n}. \tag{4.1.38}$$

4.2 Multi-loop parametrisation and Symanzik polynomials

For the Symanzik polynomials you can see Bogner and Weinzierl's article, where it's used a very similar notation.

It is possible to find a general parametrisation for any multi-loop graph in which we can perform the integration in k_i^μ, but leave integrals over Feynman/Schwinger parameters. Let's consider a 2-loop graph with n denominator defined through:

$$D_i = D(q_i, m_i) = q_i^2 - m_i^2 \tag{4.2.1}$$
$$q_i = \lambda_{ij} k_j^\mu + \sigma_{il} p_l^\mu \tag{4.2.2}$$

where k_j^μ are the loop momenta ($j = 1, 2$) and p_k^μ are the indipendent external momentum. We define in general:

$$I_n^{(2)[d]} = \int_{k_1} \int_{k_2} \frac{1}{\prod_{i=1}^n D_i}. \tag{4.2.3}$$

The combined argument of the exponential after introducing Schwinger parameters x_i for each D_i is $\sum_i x_i D_i$ which we may decompose according to the loop momentum structures.:

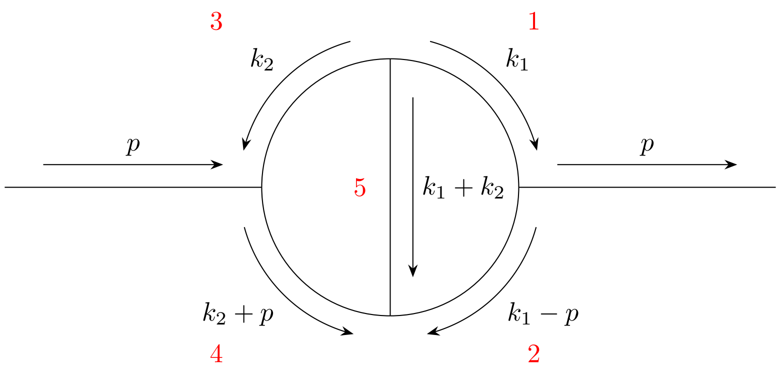
$$\sum x_i D_i = \underbrace{k_{i\mu} M_{ij} k_j^\mu}_{k^T M k} + 2 \underbrace{k_{i\mu} Q_i^\mu}_{k^T Q = Q^T k} + J \tag{4.2.4}$$

where:

- M is a geometric symmetric 2x2 matrix depending on the Schwinger parameters x_i.
- Q_i^μ is a vector linear in p_l^μ with coefficient dependent on x_i.
- J is a scalar quantity depending on momentum invariants and x_i.

All these quantities are problem dependent.

As a concrete example we can look at a two-loop propagator graph (e.g. in ϕ^3 with all equal masses $m_i = m$):



The matrix M can be written simply following the propagator dependence on each Schwinger parameter:

$$M = \begin{pmatrix} x_1 + x_2 + x_3 & x_5 \\ x_5 & x_3 + x_4 + x_5 \end{pmatrix}. \tag{4.2.5}$$

Linear dependence of k_1, k_2 cames only from propagators 2 and 4 so:

$$Q^\mu = \{-p^\mu x_2, p^\mu x_4\} \tag{4.2.6}$$

and J contains all mass dependence and p^2 terms:

$$J = -\left(\sum_{i=1}^{5} x_i\right) m^2 + (x_2 + x_4)\, p^2. \tag{4.2.7}$$

From the general form (4.2.4) we can "complete the square" in k, we do that in 2 stages: first diagonalize M, then shift it using Q.

First step: A general symmetric 2x2 matrix is:

$$M = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \tag{4.2.8}$$

completing the square in k_1 can be performed via transformation with *unit determinant*:

$$\Lambda = \begin{pmatrix} 1 & -c/a \\ 0 & 1 \end{pmatrix} \tag{4.2.9}$$

where, with $\vec{k} = (k_1, k_2)$, we have:

$$\vec{k} = \Lambda \vec{k}' \tag{4.2.10}$$

so this implies:

$$\begin{aligned} \vec{k}^T M \vec{k} &= (\Lambda \vec{k}')^T M (\Lambda \vec{k}') && (4.2.11)\\ &= (\vec{k}')^T (\Lambda^T M \Lambda) \vec{k}' && (4.2.12)\\ &= (\vec{k}')^T D \vec{k}' && (4.2.13) \end{aligned}$$

with:

$$D = \Lambda^T M \Lambda = \begin{pmatrix} a & 0 \\ 0 & \frac{\det\{M\}}{a} \end{pmatrix}. \tag{4.2.14}$$

Second step: absorb the linear terms in k_i via a transformation:

$$\vec{k}' = \vec{k}'' - \Lambda^{-1} M^{-1} \vec{Q} \tag{4.2.15}$$

so the exponential's argument after stage 1 transformation:

$$\begin{aligned} &(\vec{k}')^T D \vec{k}' + 2\vec{Q}^T \Lambda \vec{k}' + J = && (4.2.16)\\ &\quad = (\vec{k}'')^T D \vec{k}'' - (\vec{k}'')^T D \Lambda^{-1} M^{-1} \vec{Q} - (\Lambda^{-1} M^{-1} \vec{Q})^T D \vec{k}'' + \\ &\qquad + (\Lambda^{-1} M^{-1} \vec{Q})^T D \Lambda^{-1} M^{-1} \vec{Q} + \\ &\qquad + 2\vec{Q}^T \Lambda \vec{k}'' - 2\vec{Q}^T \Lambda \Lambda^{-1} M^{-1} \vec{Q} + J && (4.2.17)\\ &\quad = \vec{k}'' D \vec{k}'' + J - \vec{Q}^T M^{-1} \vec{Q}. && (4.2.18) \end{aligned}$$

Both steps can be combined so that:

$$\vec{k}'' = \Lambda\vec{k} - M^{-1}\vec{Q} \tag{4.2.19}$$

and the argument of the exponential is written as:

$$\prod x_i D_i = \vec{k}^T D\vec{k} + \frac{F}{u} \tag{4.2.20}$$

where:

$$u = \det\{M\} \tag{4.2.21}$$

which is the *first Symanzik polynomial*, and:

$$F = (J - \vec{Q}^T M^{-1}\vec{Q})u \tag{4.2.22}$$

is the *second Symanzik polynomial*, and last:

$$D = \begin{pmatrix} a & 0 \\ 0 & u/a \end{pmatrix}. \tag{4.2.23}$$

The transformation has separated the loop integral so we can perform the integration:

$$I_n^{(2)[d]} = \int \prod_{i=1}^{n} \mathrm{d}x_i \, \exp\left\{\frac{F}{u}\right\} \int_{k_1} \exp\{a k_1^2\} \int_{k_2} \exp\left\{\frac{u}{a} k_2^2\right\} \tag{4.2.24}$$

$$= \left(\frac{i}{(4\pi)^{d/2}}\right)^2 \int \prod_{i=1}^{n} \mathrm{d}x_i \, \exp\left\{\frac{F}{u}\right\} u^{-d/2} \tag{4.2.25}$$

where we used the result (4.1.19) two times. We may change from variables x_i (in $0,\infty$) to α_i (in $0,1$) by inserting:

$$1 = \int_0^\infty \mathrm{d}\lambda \, \delta\left(\lambda - \sum_{i=1}^{n} x_i\right) \tag{4.2.26}$$

and rescaling each $x_i = \lambda\alpha_i$, so:

$$I_n^{(2)[d]} = \left(\frac{i}{(4\pi)^{d/2}}\right)^2 \int_0^1 \prod_{i=1}^{n} \mathrm{d}\alpha_i \, \delta\left(1 - \sum_{i=1}^{n} \alpha_i\right) \times$$

$$\times \int \frac{\mathrm{d}\lambda}{\lambda} \lambda^{n-d} \exp\left\{\lambda \frac{F}{u}\right\} u^{-d/2} \tag{4.2.27}$$

using the change $z = -\lambda \frac{F}{u}$

$$= \left(\frac{i}{(4\pi)^{d/2}}\right)^2 \Gamma(n-d) \int_0^1 \prod_{i=1}^{n} \mathrm{d}\alpha_i \, \delta\left(1 - \sum_{i=1}^{n} \alpha_i\right) \left(-\frac{F}{u}\right)^{d-n} u^{-d/2} \tag{4.2.28}$$

$$= \left(\frac{i}{(4\pi)^{d/2}}\right)^2 \Gamma(n-d) \int_{\alpha_i} \delta\left(1 - \sum \alpha_i\right) \frac{u^{n-3d/2}}{(-F)^{n-d}}. \tag{4.2.29}$$

By following the same procedure we can derive a general multi-loop expression valid for arbitrary powers of the denominator.

Exercise 4. Show that a general L-loop Feynman integral with n internal lines may be written as:

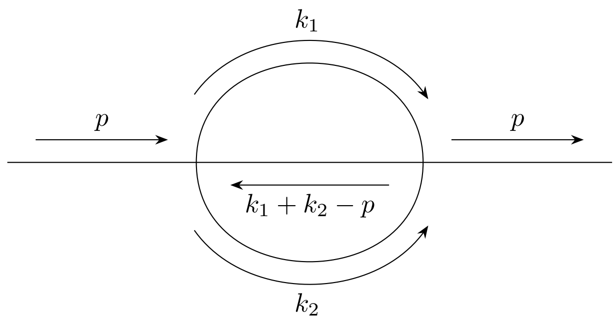
$$I_n^{(L)[d]}[\nu_1,\ldots,\nu_n] = \int \prod_{l=1}^{L} \frac{\mathrm{d}^d k_l}{(2\pi)^d} \frac{1}{\prod_{j=1}^{n} D(q_j,m_j)^{\nu_j}} \tag{4.2.30}$$

$$= \left(\frac{i}{(4\pi)^{d/2}}\right)^L \frac{\Gamma\left(\sum_{i=1}^{n}\nu_i - L\frac{d}{2}\right)}{\prod_{i=1}^{n}\Gamma(\nu_i)} \times$$

$$\times \int_{\alpha_i} \delta\left(1-\sum \alpha_i\right) \frac{u^{n-(L+1)d/2}}{(-F)^{n-dL/2}} \tag{4.2.31}$$

where u and F are the Symanzik polynomials. More details on Bogner and Weinzierl's article *Feynman Integrals*.

Let's see an **example**. The *sunrise*/*sunset* integral (massless propagator):



where we have:

$$M = \begin{pmatrix} x_1+x_2 & x_3 \\ x_3 & x_2+x_3 \end{pmatrix} \quad ; \quad \vec{Q}^{\,\mu} = \begin{pmatrix} -x_3p^\mu & -x_3p^\mu \end{pmatrix} \tag{4.2.32}$$

$$J = x_3p^2 \tag{4.2.33}$$

so:

$$u = x_1x_2 + x_1x_3 + x_2x_3 \tag{4.2.34}$$

$$F = p^2x_1x_2x_3 \tag{4.2.35}$$

that implies:

$$I_{3,\text{sunrise}}^{(2)[4-2\epsilon]} = \frac{\Gamma(-1+2\epsilon)}{(4\pi)^{4-2\epsilon}} \int \prod_{i=1}^{3} \mathrm{d}x_i\, \delta(1-x_1-x_2-x_3)(-p^2x_1x_2x_3)^{1-2\epsilon} \times$$

$$\times\, (x_1x_2+x_2x_3+x_1x_3)^{-3(1-\epsilon)} \tag{4.2.36}$$

where, if we want, we can use:

$$\Gamma(-1+2\epsilon) = \frac{-\Gamma(1+2\epsilon)}{2\epsilon(1-2\epsilon)}. \tag{4.2.37}$$

Direct integration in Feynman parameter space requires a bit of thought. Although, the $1/\epsilon$ pole is quite easy since we may expand the integrand around $\epsilon = 0$. The kinematic dependence can also be factored out:

$$I^{(2)[4-2\epsilon]}_{3,\text{sunrise}} = \left(\frac{C_\Gamma}{(4\pi)^{2-\epsilon}}\right)^2 (-p^2)^{1-2\epsilon} \hat{I}^{(2)[4-2\epsilon]}_{3,\text{sunrise}} \tag{4.2.38}$$

where we have the coefficient:

$$C_\Gamma = \frac{\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \tag{4.2.39}$$

and the normalized integral is given by:

$$\begin{aligned}
\hat{I}^{(2)[4-2\epsilon]}_{3,\text{sunrise}} &= -\frac{\Gamma(1+2\epsilon)}{2\epsilon(1-2\epsilon)C_\Gamma^2}\int_{x_i} \delta(1-\sum x_i)(x_1x_2x_3)^{1-2\epsilon}\times \\
&\quad \times (x_1x_2+x_1x_3+x_2x_3)^{-3(1-\epsilon)} && (4.2.40)\\
&= -\frac{1}{4\epsilon} - \frac{13}{8} - \frac{115}{16}\epsilon - \left(\frac{865}{32} - \frac{3}{2}\zeta_3\right)\epsilon^2 - \\
&\quad - \left(\frac{5971}{64} - \frac{39}{4}\zeta_3 - \frac{\pi^4}{40}\right)\epsilon^3 + \mathcal{O}(\epsilon^4). && (4.2.41)
\end{aligned}$$

One quick method to check this is numerical integration with many digits plus fitting to an ansatz of trascendental numbers (PSLQ algorithm).

4.3 Passarino-Veltman tensor reduction

We have already seen how to deal with higher rank numerators in Feynman or Schwinger parameter space. There is a more direct method to perform the reduction in momentum space and to determine a basis of Feynman integral at 1-loop. The principles are to:

a. Identify a *basis of possible tensor structures* after integration using p_i^μ external momentum and $g^{\mu\nu}$ metric tensor.

b. Contract the tensor integrand (in d-dimension) with each element of the tensor basis to construct a linear system of equations for the *form factors*.

Example: rank 1 bubble. We have:

$$\begin{aligned}
I_2^{(1)[d]}[k^\mu] &= \int_k \frac{k^\mu}{D(k,m)D(k-p,m)} && (4.3.1)\\
&= b_1 p^\mu && (4.3.2)
\end{aligned}$$

where we applied the step a indicating b_1 as form factor and p^μ the tensor basis. Note that we use $[k^\mu]$ as a slightly more general notation indicating the numerator.

If we apply the step b we have to contract with p^μ (basis vector):

$$
\begin{aligned}
I_2^{(1)[d]}[k \cdot p] &= b_1 p^2 && (4.3.3)\\
&= \int_k \frac{k \cdot p}{D(k,m)D(k-p,m)} && (4.3.4)\\
&= \int_k \frac{1}{2}\frac{D(k,m) - D(k-p,m) + p^2}{D(k,m)D(k-p,m)} && (4.3.5)\\
&= \frac{1}{2}\Big(\int_k \frac{1}{D(k-p,m)} - \int_k \frac{1}{D(k,m)} + \\
&\qquad + p^2 \int_k \frac{1}{D(k,m)D(k-p,m)}\Big) && (4.3.6)
\end{aligned}
$$

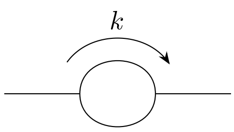
where in the first step we completed the square and we can notice that the first and second term, in the fourth line, cancel out after a shift integration $k \to k + p$ in the first one. Finishing the calculation we have:

$$
I_2^{(1)[d]}[k \cdot p] = \frac{1}{2} p^2 I_2^{(1)[d]}[1] \tag{4.3.7}
$$

so we find:

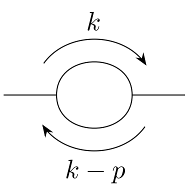
$$
b_1 = \frac{1}{2} I_2^{(1)[d]}[1]. \tag{4.3.8}
$$

We can also write these steps graphically:

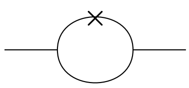
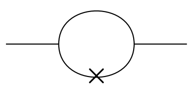


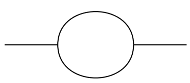
$$
[k^\mu] = I_2^{(1)[d]}[k^\mu] \tag{4.3.9}
$$

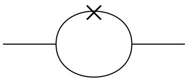
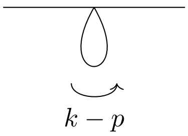
then:

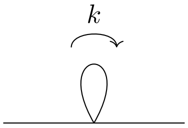


$$
[k \cdot p] = b_1 p^2 \tag{4.3.10}
$$

$= \frac{1}{2}$  $[1] - \frac{1}{2}$  $[1]$

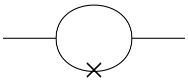
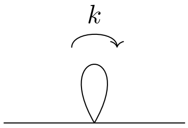
$+ \frac{p^2}{2}$  $[1]$ (4.3.11) where $-\!\!\times\!\!-$ indicates a cancelled propagator. We also have:

 [1] = 

[1] = 

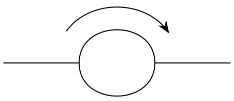
[1] (4.3.12)

where to write the last loop we shifted the loop momentum. We have also:

 [1] = 

[1] (4.3.13)

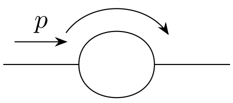
so, cancelling the other two contribution, we obtain:

$b_1 = \frac{1}{2}$ 

[1]. (4.3.14)

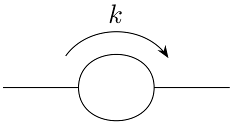
The principle is straightforward to apply to higher rank and higher multiplicity integrals though the expressions can become quite lengthly.

Example: rank 2 tensor bubble. We write the tensor basis:

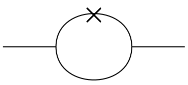
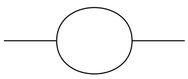


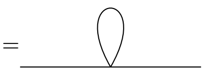
$[k^\mu k^\nu] = b_{00} g^{\mu\nu} + b_{11} p^\mu p^\nu$ (4.3.15)

so we need to determine two form factor, and now contract with:

$g_{\mu\nu}$: 

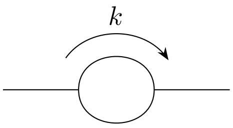
$[k^2] = d b_{00} + b_{11} p^2$ (4.3.16)

=  [1] $+m^2$  [1]

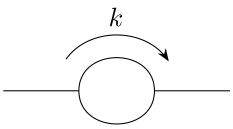
=  [1]

(4.3.17)

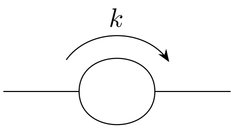
and the second element of the basis (contracting $p_\mu p_\nu$):



$$[(k\cdot p)^2] = b_{00}p^2 + b_{11}p^4 \tag{4.3.18}$$

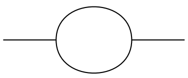
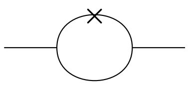
$$=$$ 

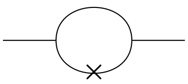
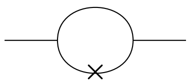
$$\left[\left(\frac{1}{2}(D(k,m) - D(k-p,m)) + p^2\right)^2\right] \tag{4.3.19}$$

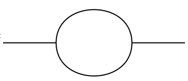
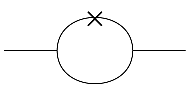
$$= \frac{1}{4}$$ 

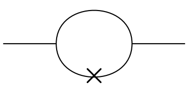
$$[D(k,m)^2 + D(k-p,m)^2 - \\ - 2D(k,m)D(k-p,m) + p^4 + 2p^2(D(k,m) - D(k-p,m))] \tag{4.3.20}$$

where if we shift k to keep on second term then cancel the last bracket, so:

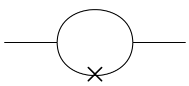
$$= \frac{1}{4}\Bigg(p^4$$  $$[1] +$$  $$[D(k,m)] +$$

$$+$$  $$[D(k-p,m)] -$$  $$[D(k,m)]\Bigg) \tag{4.3.21}$$

$$= \frac{1}{4}\Bigg(p^4$$  $$[1] +$$  $$[D(k,m)] -$$

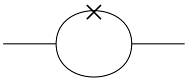
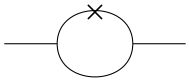
$$- 2$$  $$[D(k,m)]\Bigg) \tag{4.3.22}$$

where we simply expanded the bracket $[\cdot]$. We define:

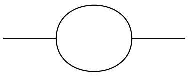
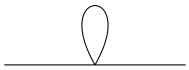


$$[D(k,m)] = \int_k \frac{k^2 - m^2}{k^2 - m^2} \equiv 0 \tag{4.3.23}$$

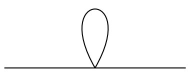
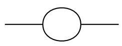
so we can perform the integral:


$$[D(k,m)] = \int_k \frac{k^2-m^2}{(k-p)^2-m^2} \tag{4.3.24}$$
$$= \int_k \frac{(k+p)^2-m^2}{k^2-m^2} \tag{4.3.25}$$
$$= \int_k \frac{k^2-m^2}{k^2-m^2} + 2\int_k \frac{k\cdot p}{k^2-m^2} + p^2 \int_k \frac{1}{k^2-m^2} \tag{4.3.26}$$
$$= p^2$$

$$[1] \tag{4.3.27}$$

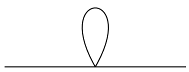
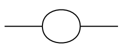
that implies:

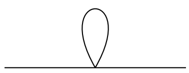
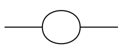
$$\frac{1}{4}p^4$$

$$[1] + \frac{1}{2}p^2$$

$$[1] = p^2 b_{00} + p^4 b_{11}. \tag{4.3.28}$$

So we have determined a system of equations for b_{00} and b_{11}:

$$\begin{pmatrix} d & p^2 \\ 1 & p^2 \end{pmatrix} \begin{pmatrix} b_{00} \\ b_{11} \end{pmatrix} = \begin{pmatrix} 1 & m^2 \\ 1/2 & 1/4p^2 \end{pmatrix} \left(\begin{array}{c} \text{[tadpole]}\ [1] \\ \text{[bubble]}\ [1] \end{array} \right) \tag{4.3.29}$$
 [1]
 [1]

where to determine the coefficient we need to invert the matrix:

$$\begin{pmatrix} b_{00} \\ b_{11} \end{pmatrix} = \frac{1}{p^2(d-1)} \begin{pmatrix} p^2 & -p^2 \\ -1 & d \end{pmatrix} \begin{pmatrix} m^2 \\ \frac{1}{2}\frac{1}{4}p^2 \end{pmatrix} \tag{4.3.30}$$
 [1]
 [1]

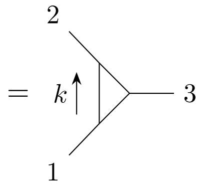
$$= \frac{1}{2(d-1)} \begin{pmatrix} 1 & -\frac{p^2}{2}\left(1-\frac{4m^2}{p^2}\right) \\ -\frac{(d-2)}{p^2} & \frac{1}{2}\left(d-\frac{4m^2}{p^2}\right) \end{pmatrix} \tag{4.3.31}$$
 [1]
 [1]
.

Worked example Perform the tensor reduction for rank 1,2 and 3 triangle integral in the massless limit with kinematics:

$$p_1^2 = p_2^2 = 0 \quad ; \quad p_3^2 \neq 0. \tag{4.3.32}$$

We define:

$$I_3^{(1)[d]}[\mathcal{N}] = \int_k \frac{\mathcal{N}}{D(k,0)D(k+p_2)D(k-p_1)} \tag{4.3.33}$$

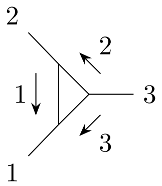


(4.3.34)

where we have the three cases:

$$\mathcal{N} = \begin{pmatrix} k^{\mu_1} & k^{\mu_1}k^{\mu_2} & k^{\mu_1}k^{\mu_2}k^{\mu_3} \end{pmatrix}. \tag{4.3.35}$$

The solution is:



with:

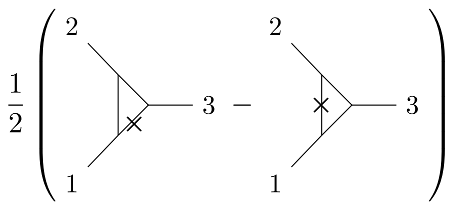
$$D = \left\{k^2\ ,\ (k+p)^2\ ,\ (k-p_1)^2\right\} \tag{4.3.36}$$

at *rank 1* we have the tensor basis $\{p_1^\mu, p_2^\mu\}$ with form factors $\{c_1, c_2\}$, so we form the system of equation form:

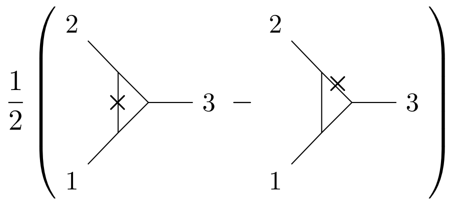
$$k \cdot p_1 = \frac{1}{2}\left(k^2 - (k-p_1)^2 + p_1^2\right) \tag{4.3.37}$$

$$k \cdot p_2 = \frac{1}{2}\left((k+p_2)^2 - k^2 - p_2^2\right) \tag{4.3.38}$$

in addition take $p_i^2 = 0$:

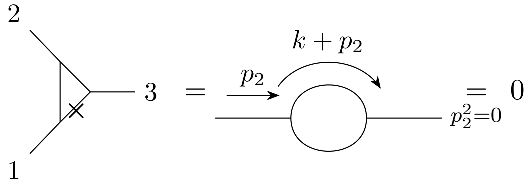


$= c_2\, p_1 \cdot p_2$ (4.3.39)



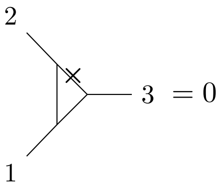
$= c_1\, p_1 \cdot p_2.$ (4.3.40)

Let us look at this integral (scaleless) to start:



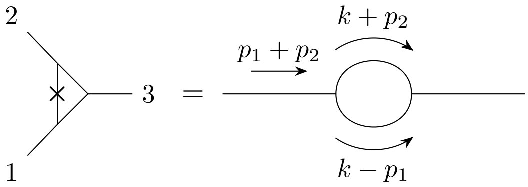
(4.3.41)

but we also have for symmetry:



(4.3.42)

so another scaleless integral. We can see:

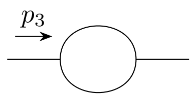


$$= I_2^{(1)[d]}((p_1+p_2)^2) \tag{4.3.43}$$

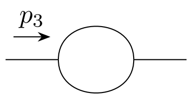
where:

$$(p_1+p_2)^2 = 2p_1 \cdot p_2 = p_3^2 = s_3 \tag{4.3.44}$$

which is invariant. In this way we get:

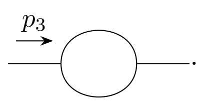
$$-\frac{1}{2}$$ 

$$= \frac{1}{2} s_3 c_2 \tag{4.3.45}$$

$$\frac{1}{2}$$ 

$$= \frac{1}{2} s_3 c_1 \tag{4.3.46}$$

so we can write:

$$\{c_1, c_2\} = \frac{1}{s_3}\{1, -1\}$$ 

. (4.3.47)

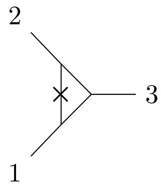
At *rank 2* we have the tensor basis:

$$\left\{ g^{\mu_1\mu_2} \ , \ p_1^{\mu_1} p_1^{\mu_2} \ , \ p_2^{\mu_1} p_2^{\mu_2} \ , \ \frac{1}{2}(p_1^{\mu_1} p_2^{\mu_2} + p_1^{\mu_2} p_2^{\mu_1}) \right\} \tag{4.3.48}$$

with form factors:

$$\{c_{00} \ , \ c_{11} \ , \ c_{22} \ , \ c_{12}\}. \tag{4.3.49}$$

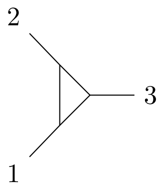
Contracting with $g_{\mu_1\mu_2}$ is quite easy:



$$= dc_{00} + p_1 \cdot p_2 c_{12} \tag{4.3.50}$$

$$= dc_{00} + \frac{1}{2} s_3 c_{12} \tag{4.3.51}$$

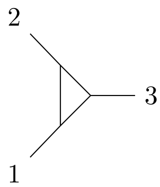
contracting with $p_{1\mu_1} p_{1\mu_2}$ gives:



$$[(k \cdot p_1)^2] = (p_1 \cdot p_2)^2 c_{22} \tag{4.3.52}$$

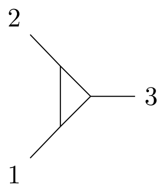
$$= \frac{1}{4} s_2^3 c_{22} \tag{4.3.53}$$

$$=$$



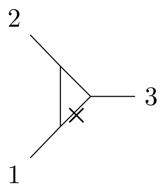
$$\left[\frac{1}{4}(D_3 - D_1)^2\right] \tag{4.3.54}$$

$$=$$

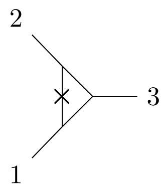


$$\left[\frac{1}{4}(D_3^2 + D_1^2 - 2D_1 D_3)\right] \tag{4.3.55}$$

$$=$$

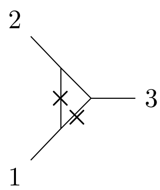


$$\left[\frac{1}{4}(k + p_1)^2\right] +$$



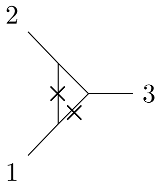
$$\left[\frac{1}{4}k^2\right] -$$

$$-\frac{1}{2}$$



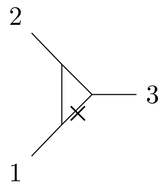
$$[1] \tag{4.3.56}$$

but the last graph is scaleless:



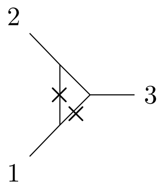
$$= \int_k \frac{1}{D_2} = \int_k \frac{1}{(k-p_2)^2} = \int_k \frac{1}{k^2} = 0. \tag{4.3.57}$$

We have:

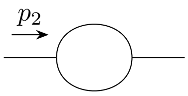


$$\left[\frac{1}{4}k^2 + \frac{1}{2}k \cdot p_1 + \underbrace{\frac{1}{4}p_1^2}_{} = 0\right]$$

$$=$$

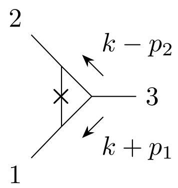


$$+ \frac{1}{2}p_1^\mu$$

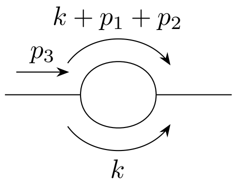


$$[k_\mu] = 0 \tag{4.3.58}$$

because both of the diagrams are scaleless. We can also see:

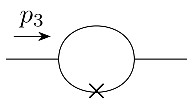


$$\left[\frac{1}{4}k^2\right] =$$

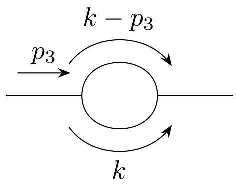


$$\left[\frac{1}{4}(k+p_2)^2\right] \tag{4.3.59}$$

$$= \frac{1}{4}$$

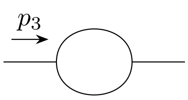


$$+ \frac{1}{2}p_2^\mu$$



$$[k_\mu] \tag{4.3.60}$$

$$= \frac{1}{4}p_2 \cdot p_3$$



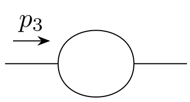
$$[1] \tag{4.3.61}$$

where we saw a rank 1 bubble, ad we have:

$$2p_2 \cdot p_3 = (p_2 + p_3)^2 - s_3 = \underbrace{p_1^2}_{=0} - s_3 \tag{4.3.62}$$

so we get:

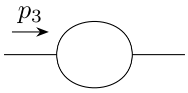
$$\frac{s_3}{8}$$



$$= \frac{1}{4}s_3^2 c_{22}. \tag{4.3.63}$$

The contraction with $p_{2\mu_1} p_{2\mu_2}$ is similar and it gives us:

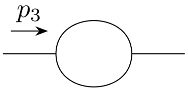
$$\frac{s_3}{8}$$



$$= \frac{1}{4} s_3^2 c_{11}. \tag{4.3.64}$$

The contraction with $\frac{1}{2}(p_{1\mu_1} p_{2\mu_2} + p_{1\mu_2} p_{2\mu_3})$ gives us:

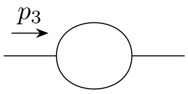
$$\frac{s_3}{8}$$



$$= \frac{1}{2} s_3^2 c_{00} + \frac{1}{8} s_3^2 c_{12}. \tag{4.3.65}$$

We can see all the equation as a matrix equation:

$$\mathcal{M} \cdot \vec{c} = \vec{b} \tag{4.3.66}$$



where:

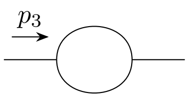
$$\vec{c} = \begin{pmatrix} c_{00} \\ c_{11} \\ c_{22} \\ c_{33} \end{pmatrix} \qquad ; \qquad \vec{b} = \begin{pmatrix} 1 \\ s_3/8 \\ s_3/8 \\ s_3/8 \end{pmatrix} \tag{4.3.67}$$

and the matrix:

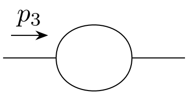
$$\mathcal{M} = \begin{pmatrix} d & 0 & 0 & s_3/2 \\ 0 & 0 & s_3^2/4 & 0 \\ 0 & s_3^2/4 & 0 & 0 \\ s_3/2 & 0 & 0 & s_3^2/8 \end{pmatrix}. \tag{4.3.68}$$

We can see we get:

$$\vec{c} = \begin{pmatrix} \frac{1}{2(d-2)} \\ -\frac{1}{2s_3} \\ -\frac{1}{2s_3} \\ \frac{d-4}{(d-2)s_3} \end{pmatrix} \times \tag{4.3.69}$$



where we can remember:



$$= I_2^{(1)[d]}(s_3). \tag{4.3.70}$$

At *rank 3* the algebra is better implemented on a computer. The final result however is relatively simple though the rank 2 bubble reduction formula is required at intermediate stages. The tensor basis is (6 elements):

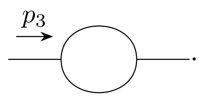
$$\begin{aligned} \vec{T} \equiv \{ & g^{\mu_1\mu_2} p_1^{\mu_3} + \text{cyclic} \, , g^{\mu_1\mu_2} p_2^{\mu_3} + \text{cyclic} \, , \\ & p_1^{\mu_1} p_1^{\mu_2} p_1^{\mu_3} \, , \, p_2^{\mu_1} p_2^{\mu_2} p_2^{\mu_3} \, , \, p_1^{\mu_1} p_1^{\mu_2} p_2^{\mu_3} + \text{cyclic} \, , \, p_1^{\mu_1} p_2^{\mu_2} p_2^{\mu_3} + \text{cyclic} \} \end{aligned} \tag{4.3.71}$$

in the only 1 basis integral as before, the form factors vector is:

$$\vec{c} = \{c_{001} \ , \ c_{002} \ , \ c_{111} \ , \ c_{222} \ , \ c_{112} \ , \ c_{122}\}. \tag{4.3.72}$$

Solving the system results in:

$$\vec{c} = \frac{1}{4(d-1)} \left\{ -1 \ , \ 1 \ , \ \frac{d}{s_3} \ , \ \frac{-d}{s_3} \ , \ \frac{-(d-4)}{s_3} \ , \ \frac{d-4}{s_3} \right\} \times \tag{4.3.73}$$



4.4 Clifford algebra in d-dimension

There are some subleties in the extension of the Clifford algebra to $d = 4-2\epsilon$ dimension and the representations of the γ matrices.

Since we will deal with QED and QCD which are parity symmetric, the issue will not arise, but for the EW part of the Standard Model we must pay attention.

Indeed, if we generalize the dimension to $d = 4 - 2\epsilon$, we need to define what are spinors in d dimensions. The spinors, as we learnt, are defined from the Clifford algebra that in 4 dimension, is:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1} \tag{4.4.1}$$

where γ^μ are 4x4 matrices. If we extend our metric to live in d dimensions we may derive identities such as:

$$\begin{aligned}
\gamma^\mu\gamma_\mu &= \frac{1}{2}\{\gamma^\mu, \gamma_\mu\} && (4.4.2)\\
&= g^\mu_\mu\mathbb{1} && (4.4.3)\\
&= d\mathbb{1} && (4.4.4)\\
\gamma^\mu\gamma^\nu\gamma_\mu &= -\gamma^\mu\gamma_\mu\gamma^\nu + 2\gamma^\nu && (4.4.5)\\
&= (2-d)\gamma^\nu && (4.4.6)
\end{aligned}$$

which will have to be applied to the numerators of the Feynman diagrams. We don't need an explicit representation of γ^μ in order to do this, but in principle we should worry about it.

It is possible to construct explicit representations for γ^μ in arbitrary dimensions, however the role of γ_5 in defining chirality only make sense in $4d$. So, the question arise naturally: what is γ_5 in $d = 4 - 2\epsilon$?

So, now we can worry about to search the γ representations in d dimension, and we can notice that they will be also Lorentz group $SO(1, d-1)$ representation.[2] The problem, as we anticiped, is that the vectorial or tensorial

[2] In facts, the matricial representations are related to universal covering group, so the spin(d).

representation have dimensionality in function of the continuous parameter d. For example, symmetric tensor has dimension $d(d+1)/2$ and the anti-symmetric has $d(d-1)/2$ dimension. Although, the spinorial representation has dimension in function to the continuous part of d, explicitly $2^{\lfloor d/2 \rfloor}$ for integer dimension d where $\lfloor \cdot \rfloor$ is the *floor* operation which round down to the nearest integer.

$2^{\lfloor d/2 \rfloor}$ is not a continuous function of d, but a step function. In $d = 2$ the spinorial representation has dimension $2^{\lfloor 1 \rfloor} = 2$, so a couple of complex numbers; but, even in $d = 3$ we have the spinorial represantation in $2^{\lfloor 1.5 \rfloor} = 2$, notice that this is the standard QM in three dimension, where the $SU(2)$ spin represantation are given by 2x2 Pauli's matrices, and even in this case the spinors are two complex numbers. In $d = 4$ we have the spinors in $2^{\lfloor 2 \rfloor} = 4$ dimensions, indeed we have the gamma matrices.

So, in an approximative way we saw that the even and odd dimensions are quite different for the spinorial representation. In the odd case the spinorial dimension is different from the vectorial dimension.

Indeed, for $d = 2k$ (k integer) the spinorial fundamental representation's dimension is simply 2^k, and if we consider only even dimension we have a spinorial dimension as a continuous function of $k \in \mathbb{N}$. In this case the fundamental representation is given by two 2^{k-1} irreducible representation[3]. In even dimension exists, in Clifford algebra, an operator (in this case γ^5, the chirality) with who we can build the projector operator in two irreducible subspace (this operator, as we know, divides the 4-dimension representation in two bidimensional represantation).

Is possible to build the analogous of γ_5 as a product of all the different reducible representation's matrices. An example can be:

$$\gamma^{2k+1} = i\frac{(-1)^k}{(2k)!}\epsilon^{\mu_1 \dots \mu_{2k}}\gamma_{\mu_1} \dots \gamma_{\mu_{2k}} \tag{4.4.7}$$

and the projector are:

$$P_\pm \equiv \frac{\mathbb{1} \pm \gamma^{2k+1}}{2}. \tag{4.4.8}$$

Now, if we consider the case $d = 2k+1$ the spinorial representation has dimension 2^k and it's not anymore a continuous function. Now, the fundamental spinorial representation is irreducible, because it's not possible build (4.4.7) in a way that is not proportional to the identity. In odd dimension doesn't exists the chirality.

We can see some group example.

[3]Note that this is a property that we already saw in the Dirac equation: we studied that the spinors doesn't a irreducible representation of Lorentz group, but they are a composition of two opposite chirality representation.

SO(2) γ **matrices** We have $\gamma^i_{(2)}$ that are 2x2 matrices where $i = 1, 2$. In Euclidean space they satisfy:

$$\left(\gamma^i_{(2)}\right)^\dagger = \gamma^i_{(2)} \tag{4.4.9}$$

and we may choose:

$$\gamma^1_{(2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (= \sigma_1) \tag{4.4.10}$$

$$\gamma^2_{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} (= \sigma_2) \tag{4.4.11}$$

so that:

$$\left\{\gamma^i_{(2)}, \gamma^j_{(2)}\right\} = 2\delta^{ij}\mathbb{1}_{2x2}. \tag{4.4.12}$$

Are sufficient two matrices to generate the space. In even dimension we may define a *chirality* operation as:

$$\hat{\gamma}^3_{(2)} = \frac{i}{2}\left(\gamma^1_{(2)}\gamma^2_{(2)} - \gamma^2_{(2)}\gamma^1_{(2)}\right) \tag{4.4.13}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.4.14}$$

so, as the third Pauli's matrix.

SO(3) γ **matrices** The dimension of the γ represantation is $2^{\lfloor d/2 \rfloor}$ for integer dimension d. So in $3d$ we have 3, 2x2 matrices satisfying:

$$\left\{\gamma^i_{(3)}, \gamma^j_{(3)}\right\} = 2\delta^{ij}\mathbb{1}_{2x2} \tag{4.4.15}$$

where:

$$\gamma^1_{(3)} = \gamma^1_{(2)} \tag{4.4.16}$$

$$\gamma^2_{(3)} = \gamma^2_{(2)} \tag{4.4.17}$$

$$\gamma^3_{(3)} = \hat{\gamma}^3_{(2)} \tag{4.4.18}$$

so the three Pauli's matrices. However there is no notion of chirality in $3d$. Higher dimension represantation can be built from tensor products, i.e. $d = 4$ 4x4 matrices built form tensor product of 2x2 matrices.

SO(1, 1) **matrices** In $2d$ Minkowski space the γ matrices satisfy (with the usual $g^{\mu\nu} = \mathrm{diag}(1, -1)$):

$$(\gamma^\mu_{(2)})^\dagger = \begin{cases} \gamma^0_{(2)} & \mu = 0 \\ -\gamma^1_{(2)} & \mu = 1 \end{cases} \tag{4.4.19}$$

or:

$$(\gamma^{\mu}_{(2)})^{\dagger} = \gamma_{(2)\mu}. \tag{4.4.20}$$

Choosing:

$$\gamma^0_{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (= \sigma_3) \tag{4.4.21}$$

$$\gamma^1_{(2)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (= \sigma_1). \tag{4.4.22}$$

We find explicitly:

$$\left\{\gamma^{\mu}_{(2)}, \gamma^{\nu}_{(2)}\right\} = 2g^{\mu\nu}\mathbb{1}_{2x2}. \tag{4.4.23}$$

We may also define a *chirality* operation as:

$$\hat{\gamma}^3_{(2)} = \frac{1}{2}\left(\gamma^0_{(2)}\gamma^1_{(2)} - \gamma^1_{(2)}\gamma^0_{(2)}\right) \tag{4.4.24}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.4.25}$$

The operator anti-commutes with $\gamma^{\mu}_{(2)}$:

$$\left\{\gamma^{\mu}_{(2)}, \hat{\gamma}^3_{(2)}\right\} = 0. \tag{4.4.26}$$

The $3d$ case is constructed analogously with the $SO(3)$ example.

$SO(1,3)$ **matrices** The Standard Dirac matrices in the Weyl basis can be built from the $SO(1,1)$ matrices using the tensor product[4]:

$$\gamma^0_{(4)} = \hat{\gamma}^3_{(2)} \otimes \mathbb{1}_{2x2} \tag{4.4.27}$$

$$\gamma^1_{(4)} = \gamma^2_{(2)} \otimes \hat{\gamma}^3_{(2)} \tag{4.4.28}$$

$$\gamma^2_{(4)} = -i\gamma^2_{(2)} \otimes \gamma^2_{(2)} \tag{4.4.29}$$

$$\gamma^3_{(4)} = \gamma^2_{(2)} \otimes \gamma^1_{(2)} \tag{4.4.30}$$

in this way:

$$\hat{\gamma}^5_{(4)} = -i\gamma^0_{(4)}\gamma^1_{(4)}\gamma^2_{(4)}\gamma^3_{(4)} \tag{4.4.31}$$

and:

$$P_{\pm} = \frac{1 \pm \hat{\gamma}^5_{(4)}}{2}. \tag{4.4.32}$$

[4]Indeed, a way to build an explicit represantation for matrices in higher dimesion is using the tensor product of Pauli's matrices, until we obtain the correct dimension. See page 82 of M. Nebbia's notes.

4.4.1 γ matrices in dimensional regularisation

If we were to think of representing the γ in $d = 4 - 2\epsilon$ dimension as an ∞ dimension vector space we would not find a correct continuation around $d = 4$. We need γ_5 to represent *chirality* in $d = 4$. We have already saw that the notion of chirality doesn't exists in odd dimension, and it's impossible to generalize it's definition in an elegant way.

The reason why it happen is due to the definition (4.4.7), where it's present the tensor ϵ, which is related to the space-time dimension. In $d = 2, 4$ we have $\epsilon_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}$ and we have two or four indices to fill the space. Otherwise, in a generic dimension d, expecially non integer, ϵ doesn't exists. The facts that we try to extend γ_5 in a generic dimension bring us to the problem which ask us to "close" the Clifford algebra using exactly the available space-time direction. When we do that we break some of the properties of gammas.

't Hooft and Veltman, and for this reason we say *'t Hooft-Veltman* scheme, propose to keep the definition of $\hat{\gamma}^5$, the *chirality*, in $d = 4$:

$$i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{4.4.33}$$

even in other dimension. This statement bring to some stranges conseguences. Therefore we keep the dimension of the spinor space to be 4:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_{4x4} \tag{4.4.34}$$

where we have $g^{\mu\nu} = \text{diag}(1, -1, -1, \dots)$. We therefore keep the identities for $\gamma^\mu\gamma_\mu$ etc. as before.

With the definition (4.4.33) and all the relation for the gammas we have:

$$\left\{\gamma^\mu, \hat{\gamma}^5\right\} = 0 \qquad , \quad \mu = (0, 1, 2, 3) \tag{4.4.35}$$

but for $\mu > 3$ we must implement a commutation relation:

$$\left[\gamma^\mu, \hat{\gamma}^5\right] = 0 \qquad , \quad \mu > 3. \tag{4.4.36}$$

In facts, we are separating the space in two parts, the physical one, with $\mu = 0, 1, 2, 3$, and the evanishing, with $\mu > 3$. In dimensional regularization this different behaviour brings with its the chirial anomaly, which is a general property of some *chirial theory*, such as the Electro-Weak interaction.

Chapter 5

Renormalization of QED at one-loop

In this chapter, as the title says, we will see how to renormalize the QED. In particular we complete renormalization of QED at one-loop though renormalization of A_μ, ψ, e and m. We will see the quantum corrections to EM interaction and we will see $g-2=\alpha/(2\pi)$, so the *anomalous magnetic moment*. Also, we'll see the quantum corrections to electric potential (Uehling potential) and the charge screening. Last but not least we will see how gauge invariance, $Z_1=Z_2$ and the Ward-Takahashi identity are related each orther.

5.1 Lagrangian, Feynman rules and counter terms

Let's see a brief rewiew of some things we already know before jump into the renormalization of QED. We consider a general R_ξ *covariant gauge.*

5.1.1 Bare Lagrangian and gauge invariance

We know the lagrangian of our theory:

$$\mathcal{L}_{QED}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\overline{\psi}(i\not{\mathcal{D}}-m)\psi+\mathcal{L}_{gf} \tag{5.1.1}$$

where we have:

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu \tag{5.1.2}$$

$$\mathcal{D}^\mu=\partial^\mu+ieA^\mu \tag{5.1.3}$$

and we know is invariant under local $U(1)$ rotation:

$$\psi \;\longrightarrow\; \psi'=e^{i\theta(x)}\psi \tag{5.1.4}$$

$$A_\mu \;\longrightarrow\; A'_\mu=A_\mu-\frac{1}{e}\partial_\mu\theta(x). \tag{5.1.5}$$

The covariant gauge fixing term is:

$$\mathcal{L}_{gf} = \frac{1}{2\xi}(\partial_\mu A^\mu)^2. \tag{5.1.6}$$

To derive the form of the photon propagator we take the terms in $\mathcal{L}_{QED}$ with 2 photon fields A^μ:

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{\xi}(\partial_\mu A^\mu)^2 = -\frac{1}{2}\left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu + \frac{1}{\xi}\partial_\mu A^\mu \partial_\nu A^\nu\right) \tag{5.1.7}$$

$$= \frac{1}{2}A^\mu \left(\Box g_{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial_\mu \partial_\nu\right) A^\nu + \text{surface term} \tag{5.1.8}$$

where we have used integration by parts and removed surface terms from the $\mathcal{L}$. The Fourier tranformation of (5.1.8) gives us:

$$\frac{1}{2}\tilde{A}^\mu \mathcal{M}_{\mu\nu}\tilde{A}^\nu \tag{5.1.9}$$

where:

$$\mathcal{M}_{\mu\nu} = -p^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right)p_\mu p_\nu. \tag{5.1.10}$$

The photon propagator $D_{\mu\nu}$ is defined by:

$$D_{\mu\nu}(p^2, \xi)\mathcal{M}^\nu{}_\sigma(p^2, \xi) = i g_{\mu\sigma} \tag{5.1.11}$$

so, as the inverse operator of $\mathcal{M}$. We solve (5.1.11) for $D_{\mu\nu}$ by first providing a tensor decomposition:

$$D_{\mu\nu}(p^2, \xi) = \frac{\alpha}{p^2}\left(-g_{\mu\nu} + \beta\frac{p_\mu p_\nu}{p^2}\right) \tag{5.1.12}$$

which can be inserted into (5.1.11) to find:

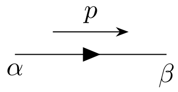
$$\alpha = i \qquad , \qquad \beta = 1 - \xi. \tag{5.1.13}$$

We need to add contour deformation to avoid the pole on the real axis, so the final expression is:

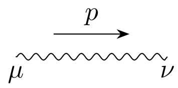
$$D_{\mu\nu}(p^2, \xi) = \frac{1}{p^2 + i\emptyset^+}\left(-g_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2}\right). \tag{5.1.14}$$

5.1.2 Feynman rules

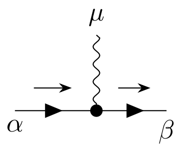
In this section we see the Feynman rules of the theory. We have the progators:



$$= \frac{i(\not{p}+m)_{\beta\alpha}}{p^2-m^2+i\not{0}^+} \tag{5.1.15}$$



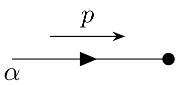
$$= \frac{i}{p^2+i\not{0}^+}\left(-g_{\mu\nu}+(1-\xi)\frac{p_\mu p_\nu}{p^2}\right) \tag{5.1.16}$$



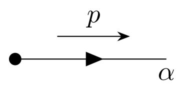
$$= -ie\gamma^{\mu}_{\alpha\beta} \tag{5.1.17}$$

plus the rules for external spinors and polarisations:

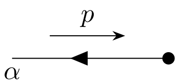
initial ; final (5.1.18)



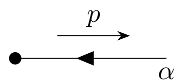
$= u_{\alpha,s}(p,m)$;



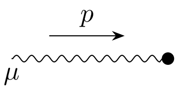
$= \overline{u}_{\alpha,s}(p,m)$ (5.1.19)



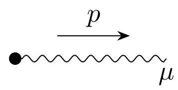
$= \overline{v}_{\alpha,s}(p,m)$;



$= v_{\alpha,s}(p,m)$ (5.1.20)

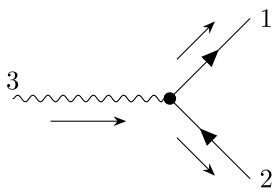


$= \epsilon_\mu(p)$;



$= \epsilon^*_\mu(p).$ (5.1.21)

For example we have the diagram:



$$= -ie\overline{u}_{s_1}(p_1,m)\not{\epsilon}(p_3)v_{s_2}(p_2,m). \tag{5.1.22}$$

5.1.3 Superficial degree of divergence

We know that the expression of the superficial degree of divergence is theory dependent, so recalling how we derived (3.1.1), where d was the dimension, L the loop order and n the number of internal lines; we may derive a similar form for a graph in QED, counting the momentum power in the loop integral:

$$\omega(\Gamma_{QED}) = dL - 2P_\gamma - P_e \tag{5.1.23}$$

where P_γ and P_e are the number of internal photons and electrons/positrons respectively.

We may express $\omega(\Gamma_{QED})$ only in terms of external photon and electron multiplicity, E_γ and E_e, using the expressions:

$$L = P_e + P_\gamma - V + 1 \tag{5.1.24}$$

and:

$$\begin{aligned} V &= 2P_\gamma + E_\gamma && (5.1.25)\\ &= P_e + \frac{1}{2}E_e && (5.1.26)\end{aligned}$$

just to write:

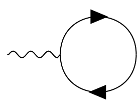
$$\omega(\Gamma_{QED}) = d + \left(\frac{1-d}{2}\right)E_e + \left(\frac{2-d}{2}\right)E_\gamma + \left(\frac{d-4}{2}\right)V \tag{5.1.27}$$

so in $4d$ is simply:

$$\omega(\Gamma_{QED}) = 4 - \frac{3}{2}E_e - E_\gamma. \tag{5.1.28}$$

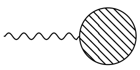
We can notice that this expression could be obtained via the analysis of operator dimension inside the loop-integral (like the expression (3.1.14)). Using the expression (5.1.28) we can see the divergent graph in the table 5.1.

In QED the treatment of the tadpole graph is different, and more straight-forward, than in ϕ^3 where it was necessary (and sufficient) to re-define the vacuum expectation value. There we can show the graph is zero e.g. at 1-loop:



$$\begin{aligned} \mu \quad &= -ie\,\mathrm{Tr}\left\{\gamma^\mu \int_k \frac{\not{k}+m}{k^2-m^2}\right\} && (5.1.29)\\ &= -ie\,\underbrace{\mathrm{Tr}\{\gamma^\mu\}}_{=0}\, m\int_k \frac{1}{k^2-m^2} - ie\,\mathrm{Tr}\{\gamma^\mu\gamma^\nu\}\underbrace{\int_k \frac{k_\nu}{k^2-m^2}}_{=0} && (5.1.30)\\ &= 0 && (5.1.31)\end{aligned}$$

thanks to the gammas' properties and the scaleless integral. This property is valid to all loop orders, so:



$$= 0. \tag{5.1.32}$$

It's very important that the photon tadpole is zero, because we know that γ has two spin polarisations (± 1), but in every cases it has to be conserved, so

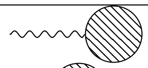
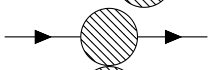
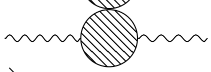
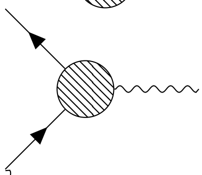
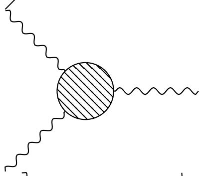
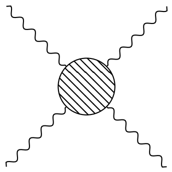
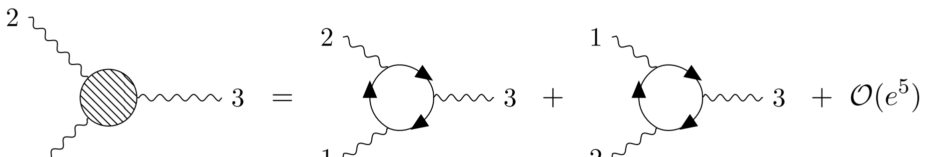
Γ	E_e	E_γ	$\omega(\Gamma_{QED})$	name
	0	1	3	tadpole
	2	0	1	electron self energy
	0	2	2	photon vacuum polarisation
	2	1	0	vertex
	0	3	1	-
	0	4	0	-

Table 5.1: Divergent diagrams of QED.

since these kind of diagrams contribute to the vacuum expectation value $\langle A \rangle$, has to be zero to not breaks the rotations invariance (choosing a privileged direction in space).

There are two other graph to worry about it - the 3 and 4 γ graphs. There aren't operators for γ self-interactions in the lagrangian, so it's not clear where to absorb the divergences. The resolution is to see that divergences cancel between contributing diagrams e.g.:

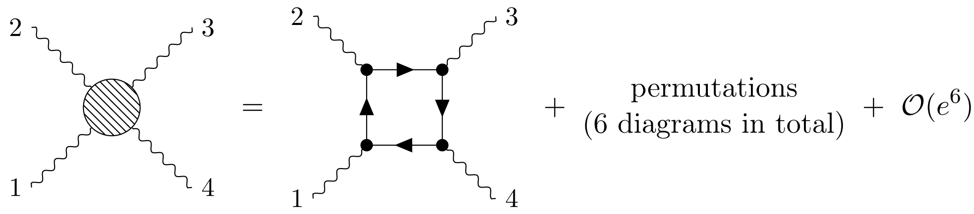


(5.1.33)

$$= 0 \tag{5.1.34}$$

where with $\mathcal{O}(e^5)$ we indicate the higher loops. The contributes cancel out because, if we write explicitly the integral, we can see that the denominators are the same (we have the same loop-momentum, in opposite directions, but with a power of two), while the numerator has a sign difference.

For the 4γ case we have:

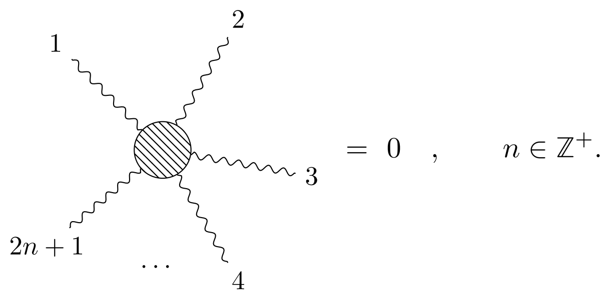


(5.1.35)

$$= \text{finite}. \tag{5.1.36}$$

The 3γ amplitude result is a special case of a more general theorem:

Theorem 1 (Furry's theorem) *Correlation functions and amplitudes for odd numbers of photons are zero:*



(5.1.37)

This theorem follows using *charge conjugation symmetry* and in facts it tells us that the correlation functions with odd external photons are zero because they generate a lots of diagrams, which contributions cancel each orther. There is a proof of this theorem in M. Nebbia's notes at page 111.

5.1.4 Renormalization constants and counterterms

In terms of bare fields, couplings and masses we have:

$$\mathcal{L}_{QED} = \frac{1}{2} A_0^\mu \left(g_{\mu\nu} \Box - \left(1 - \frac{1}{\xi_0} \right) \partial_\mu \partial_\nu \right) A_0^\nu + \\ + i\overline{\psi}_0 \not{\partial} \psi_0 - m_0 \overline{\psi}_0 \psi_0 - e_0 \overline{\psi}_0 \not{A}_0 \psi_0. \tag{5.1.38}$$

We relate bare quantities to renormalized ones using the standard procedure:

$$\psi_0 = \sqrt{Z_\psi} \psi_R \tag{5.1.39}$$

$$A_0^\mu = \sqrt{Z_A} A_R^\mu \tag{5.1.40}$$

$$m_0 = Z_m m \tag{5.1.41}$$

$$e_0 = Z_e e_R \mu_R^\epsilon = \frac{Z_1}{Z_\psi \sqrt{Z_A}} e_R \mu_R^\epsilon \tag{5.1.42}$$

$$\xi_0 = \frac{Z_A}{Z_\xi} = \xi_R \tag{5.1.43}$$

so that:

$$\mathcal{L}_{QED} = \mathcal{L}_{QED,R} + \mathcal{L}_{CT} \tag{5.1.44}$$

where we have:

$$\begin{aligned}\mathcal{L}_{CT} = &\frac{1}{2}(Z_A - 1)A_R^\mu(\Box g_{\mu\nu} - \partial_\mu\partial_\nu)A_R^\nu + \\ &+ \frac{1}{2}\underbrace{(Z_\xi - 1)}_{\xi_R} A_R^\mu \partial_\mu \partial_\nu A_R^\nu + \\ &+ i(Z_\psi - 1)\overline{\psi}_R \partial\!\!\!/ \psi_R - (Z_m Z_\psi - 1)m_R \overline{\psi}_R \psi_R - \\ &- (Z_1 - 1)e_R \mu_R^\epsilon \overline{\psi}_R A\!\!\!/_R \psi_R \end{aligned} \tag{5.1.45}$$

and the term $\mathcal{L}_{QED,R}$, which is invariant under gauge rotations:

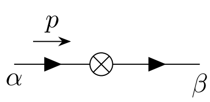
$$\psi_R \longrightarrow \psi'_R = e^{i\theta(x)}\psi_R \tag{5.1.46}$$

$$A_R^\mu \longrightarrow A'^\mu_R = A_R^\mu - \frac{1}{\mu_R^\epsilon e_R}\partial^\mu\theta(x). \tag{5.1.47}$$

For the whole expression, to be invariant, requires:

$$Z_1 = Z_\psi. \tag{5.1.48}$$

With the expression (5.1.45) we can see the Feynman rules for the counter-terms:

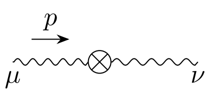


$$= i\left(p\!\!\!/\delta_\psi - m\delta_3\right)_{\beta\alpha} \tag{5.1.49}$$

where:

$$\delta_3 = Z_m Z_\psi - 1 \quad , \quad \delta_\psi = Z_\psi - 1. \tag{5.1.50}$$

We also have:

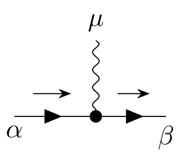


$$= -i\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right)\delta_A \tag{5.1.51}$$

where:

$$\delta_A = Z_A - 1. \tag{5.1.52}$$

The last rule is:



$$= -ie_R\mu_R^\epsilon\gamma^\mu\delta_1 \tag{5.1.53}$$

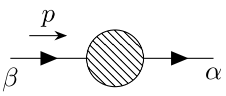
where:

$$\delta_1 = Z_1 - 1. \tag{5.1.54}$$

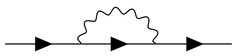
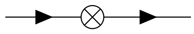
We will show later that $Z_\xi = 1$ (see section §5.2.6).

5.2 Divergent graph at 1-loop

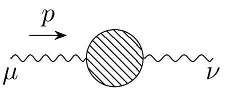
The expansions of the divergent graph to one-loop are the following. For the *electron self energy* we have:



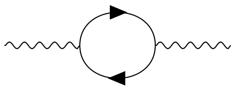
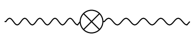
$$= \Sigma_{\beta\alpha}(p, m) \tag{5.2.1}$$

$$=$$  $+$  $+\ \mathcal{O}(e^4).$ (5.2.2)

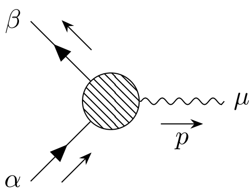
For the *vacuum polarisation* we have:



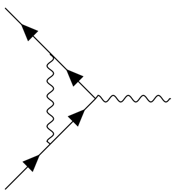
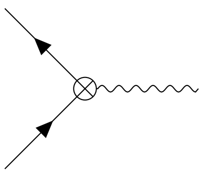
$$= \Pi_{\mu\nu}(p, m) \tag{5.2.3}$$

$$=$$  $+$  $+\ \mathcal{O}(e^4).$ (5.2.4)

The *vertex correction* is:



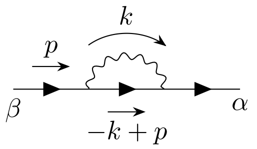
$$= \Gamma^{\mu}_{\beta\alpha}(p, m) \tag{5.2.5}$$

$$=$$  $+$  $+\ \mathcal{O}(e^5).$ (5.2.6)

Let's analize each of them individually.

5.2.1 Electron self-energy

Now we take, to start with, $\xi_R = 1$ (*Feynman gauge*). We have:



$$= (-ie_R\mu_R)^2 \int_k \gamma^\mu_{\alpha\delta} \frac{i(-\not{k}+\not{p}+m_R)_{\delta\sigma}}{D(k-p,m_R)} \gamma^\nu_{\sigma\beta} \left(\frac{-ig_{\mu\nu}}{D(k,0)}\right) \tag{5.2.7}$$

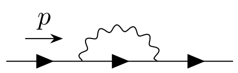
$$= e_R^2\mu_R^{2\epsilon} \int_k \frac{(\gamma^\mu(\not{k}-\not{p}-m_R)\gamma_\mu)_{\alpha\beta}}{D(k,0)D(k-p,m_R)} \tag{5.2.8}$$

$$= e_R^2\mu_R^{2\epsilon} \int_k \frac{((2-d)(\not{k}-\not{p})-dm_R)_{\alpha\beta}}{D(k,0)D(k-p,m_R)} \tag{5.2.9}$$

where we used the properties of gammas and where we can define:

$$I^{(1)[d]}_{2,m_1m_2}[\mathcal{N}] = \int_k \frac{\mathcal{N}}{D(k,m_1)D(k-p,m_2)} \tag{5.2.10}$$

so that:



$$=$$

$$= -e_R^2\mu_R^{2\epsilon}\Big\{\big[(2-d)\not{p}+dm_R\big]I^{(1)[d]}_{2,m}[1]-(2-d)I^{(1)[d]}_{2,m}[\not{k}]\Big\} \tag{5.2.11}$$

$$= -e_R^2\mu_R^{2\epsilon}\Big[\frac{(2-d)\not{p}}{2p^2}\Big((p^2+2m_R^2)I^{(1)[d]}_{2,m}[1]-I^{(1)[d]}_{1,m}\Big)+dm_RI^{(1)[d]}_{2,m}\Big] \tag{5.2.12}$$

where we have used the fact:

$$I^{(1)[d]}_{2,m}[\not{k}] = \not{p}b_1(p;0,m) \tag{5.2.13}$$

$$= \frac{\not{p}}{2p^2}\Big[I^{(1)[d]}_{1,m}[1]+(p^2-m_R^2)I^{(1)[d]}_{2,m}[1]\Big]. \tag{5.2.14}$$

The scalar integrals $d = 4-2\epsilon$ dimensional are:

$$I^{(1)[4-2\epsilon]}_{1,m}[1] = \frac{i}{(4\pi)^{2-\epsilon}}\frac{\Gamma(1+\epsilon)}{(1-\epsilon)\epsilon}(m_R^2)^{1-\epsilon} \tag{5.2.15}$$

$$= \frac{iC_\Gamma}{(4\pi)^2}m_R^2\left(\frac{1}{\epsilon}+1-\log(m_R^2)+\mathcal{O}(\epsilon)\right) \tag{5.2.16}$$

and:

$$I^{(1)[4-2\epsilon]}_{2,0m}[1] = \frac{iC_\Gamma}{(4\pi)^2}\left(\frac{1}{\epsilon}+2-\log(m_R^2)+\left(1-\frac{m_R^2}{p^2}\right)\log\left(1-\frac{p^2}{m_R^2}\right)+\mathcal{O}(\epsilon)\right) \tag{5.2.17}$$

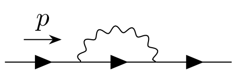
where:

$$C_\Gamma = \Gamma(1+\epsilon)(4\pi)^\epsilon. \tag{5.2.18}$$

So, using:

$$e_R^2 = 4\pi\alpha. \tag{5.2.19}$$

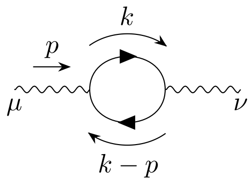
and noticing that between (5.2.16) and (5.2.17) there are some terms that cancel out, we can conclude:



$$= i\left(\frac{\alpha}{4\pi}\right)(\not{p} - 4m_R)\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0). \tag{5.2.20}$$

5.2.2 Photon vacuum polarisation

We can compute (with a minus sing for the closed fermion loop):

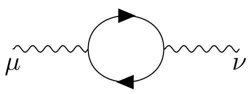


$$= -(-ie_R\mu_R)^2 \int_k \frac{\mathrm{Tr}\{\gamma^\mu i(\not{k} + m_R)\gamma^\nu i(\not{k} - \not{p} + m_R)\}}{D(k, m_R)D(k-p, m_R)} \tag{5.2.21}$$

where we have:

$$\begin{aligned}\mathrm{Tr}\{\gamma^\mu(\not{k} + m_R)\gamma^\nu(\not{k} - \not{p} + m_R)\} = 4\Big(&2k^\mu k^\nu - k^2 g^{\mu\nu} - k^\mu p^\nu - \\ &- k^\nu p^\mu + k\cdot p g^{\mu\nu} + m_R^2 g^{\mu\nu}\Big)\end{aligned} \tag{5.2.22}$$

so we need the tensor reduction for up to rank 2 bubble integrals. After applying Passarino-Veltman reduction we find:

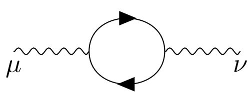


$$= -e_R^2\mu_R^{2\epsilon}\Big(-g^{\mu\nu}p^2 + p^\mu p^\nu\Big)A \tag{5.2.23}$$

with:

$$A = \frac{4}{p^2(d-1)}\left[(d-2)I_{1,m}^{(1)[d]} - \left(2m_R^2 + \frac{d-2}{2}p^2\right)I_{2,mm}^{(1)[d]}\right]. \tag{5.2.24}$$

We can see that the bubble integral is the same we used in ϕ^3 theory, so in $d = 4 - 2\epsilon$ dimensions on can show:

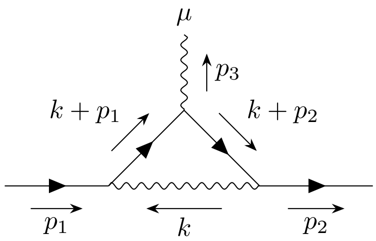


$$= \Big(-g^{\mu\nu}p^2 + p^\mu p^\nu\Big)\left(-i\frac{\alpha}{4\pi}\right)\left(-\frac{4}{3\epsilon} + \mathcal{O}(\epsilon^0)\right). \tag{5.2.25}$$

You can find the complete results, with finite terms, in Mathematica for example.

5.2.3 Vertex correction

This computation requires considerable extra work and is necessary to fix $\delta_1^{(1)}$. If we accept the result $Z_1 = Z_2$, then we can avoid the computation, but validating the result is an extremely good cross check. The one-loop diagram is:



$$= -ie_R\mu_R^{\epsilon}\overline{u}(p_2, m_R)\Gamma^{(1)\mu}u(p_1, m_R) \tag{5.2.26}$$

where we have the conservation of momentum and (in Feynman gauge):

$$\Gamma^{(1)\mu} = -i^3(-ie_R\mu_R^{\epsilon})^2 \int_k \frac{\gamma^\nu(\not{k} + \not{p}_2 + m_R)\gamma^\mu(\not{k} + \not{p}_1 + m_R)\gamma_\nu}{D(k,0)D(k+p_1, m_R)D(k+p_2, m_R)}. \tag{5.2.27}$$

A general basis for $\Gamma^\mu_{\alpha\beta}(p_1, p_2)$ would have 3 elements $\gamma^\mu_{\alpha\beta}, p_1^\mu\delta_{\alpha\beta}$ and $p_2^\mu\delta_{\alpha\beta}$. However, Γ^μ satisfies the **Ward identity**:

$$\Gamma^\mu(p_1, p_2)p_{3\mu} = \Gamma^\mu(p_1, p_2)(p_1 - p_2)_\mu = 0. \tag{5.2.28}$$

As a result, Γ^μ can be described in terms of 2 indipendent tensor structures. The vertex descrives the elettromagnetic interaction and so it is useful to use a tensor basis that exposes electric and magnetic properties (we will explore the direct connection later). We can write:

$$\Gamma^\mu(p_1, p_2) = F_1\gamma^\mu + F_2\frac{i\sigma^{\mu\nu}}{2m_R}(-p_1 + p_2)_\nu \tag{5.2.29}$$

where:

$$\sigma^{\mu\nu} = \frac{-i}{2}[\gamma^\mu, \gamma^\nu] \tag{5.2.30}$$

and F_1, F_2 are the *electric* and *magnetic structure functions* respectively.

Proof of Γ^μ decomposition We start with:

$$\overline{u}_2\Gamma^\mu u_1 = a_0\overline{u}_2\gamma^\mu u_1 + a_1p_1^\mu\overline{u}_2u_1 + a_2p_2^\mu\overline{u}_2u_1 \tag{5.2.31}$$

we can change basis to $(p_1 + p_2)_\mu$ and $(p_1 - p_2)_\mu$, so:

$$\overline{u}_2\Gamma^\mu u_1 = a_0\overline{u}_2\gamma^\mu u_1 + \tilde{a}_1(p_1 - p_2)^\mu\overline{u}_2u_1 + \tilde{a}_2(p_1 + p_2)^\mu\overline{u}_2u_1 \tag{5.2.32}$$

contracting with p_3^μ (or with $p_1 - p_2$ using the momentum conservation law), that means that we are forcing the Ward identity:

$$\overline{u}_2\Gamma \cdot p_3 u_1 = -\overline{u}_2\Gamma \cdot (p_1 - p_2)u_1 \tag{5.2.33}$$
$$= -a_0\overline{u}_2(\not{p}_1 - \not{p}_2)u_1 - \tilde{a}_1(p_1 - p_2)^2\overline{u}_2 u_1 - \tilde{a}_2(p_1^2 - p_2^2)\overline{u}_2 u_1 \tag{5.2.34}$$
$$= 0 \tag{5.2.35}$$

where we used the fact that the particle are on-shell, so the first term is zero because are valid:

$$\overline{u}_2\not{p}_2 = m\overline{u}_2 \tag{5.2.36}$$
$$\not{p}_1 u_1 = m u_1 \tag{5.2.37}$$

and the third term is zero since:

$$p_1^2 = p_2^2 = m_R^2. \tag{5.2.38}$$

So we have found:

$$\tilde{a}_1 = 0. \tag{5.2.39}$$

Now we can notice that:

$$p^\mu\overline{u}_2 u_1 = p_\nu g^{\nu\mu}\overline{u}_2 u_1 \tag{5.2.40}$$
$$= \frac{1}{2}p_\nu\overline{u}_2\{\gamma^\mu, \gamma^\nu\}u_1 \tag{5.2.41}$$
$$= \frac{1}{2}\overline{u}_2\{\gamma^\mu, \not{p}\}u_1 \tag{5.2.42}$$

hence:

$$(p_1 + p_2)^\mu\overline{u}_2 u_1 = \frac{1}{2}\overline{u}_2\left\{\gamma^\mu, \not{p}_1 + \not{p}_2\right\}u_1 \tag{5.2.43}$$
$$= \frac{1}{2}\overline{u}_2\left(\gamma^\mu\not{p}_2 + \not{p}_1\gamma^\mu + 2m_R\gamma^\mu\right)u_1 \tag{5.2.44}$$

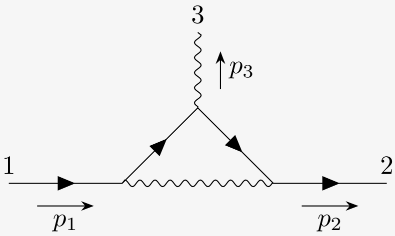
and now we can expand the magnetic tensor:

$$\overline{u}_2\left(\frac{i\sigma^{\mu\nu}}{2m_R}(-p_1 + p_2)_\nu\right)u_1 = \overline{u}_2\frac{1}{4m_R}\left[\gamma^\mu, -\not{p}_1 + \not{p}_2\right]u_1 \tag{5.2.45}$$
$$= \frac{1}{4m_R}\overline{u}_2\left(\gamma^\mu\not{p}_2 + \not{p}_1\gamma^\mu - 2m_R\gamma^\mu\right)u_1 \tag{5.2.46}$$
$$= \frac{1}{2m_R}(p_1 + p_2)^\mu\overline{u}_2 u_1 - \overline{u}_2\gamma^\mu u_1 \tag{5.2.47}$$

which completes the rearrangement. ■

The full computation of the vertex diagram is quite long. It's useful to try it in different ways. See the following exercises.

Exercise 5. The UV divergence for a triangle topology can be extracted by taking $k \to \infty$ at the integrand level. Compute the resulting integral using Feynman parameters to show that:



$$= -ie_R\mu_R^\epsilon \overline{u}_2\gamma^\mu u_1 \left(\frac{\alpha}{4\pi}\right)\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0). \tag{5.2.48}$$

Exercise 6. Perform the reduction to scalar integrals using the Passarino-Veltman method to show:

$$F_1^{(1)} = \frac{\alpha}{4\pi}C_\Gamma\left(\frac{1}{\epsilon} + \log\left(\frac{\mu_R^2}{m^2}\right) + \right.$$
$$\left. + \frac{3Q^2 - 8m^2}{Q^2\beta}\log\left(-\frac{\beta_+}{\beta_-}\right) + \hat{I}_3\right) + \mathcal{O}(\epsilon) \tag{5.2.49}$$

where:

$$\beta = \sqrt{1 - \frac{4m^2}{Q^2}} \quad , \quad \beta_\pm = \frac{1 \pm \beta}{2} \quad , \quad (-p_1 + p_2)^2 = Q^2 \tag{5.2.50}$$

and:

$$I_{3,0mm}^{(1)[4-2\epsilon]} = \frac{iC_\Gamma}{(4\pi)^2}\frac{1}{4p_1 \cdot p_2}\hat{I}_3. \tag{5.2.51}$$

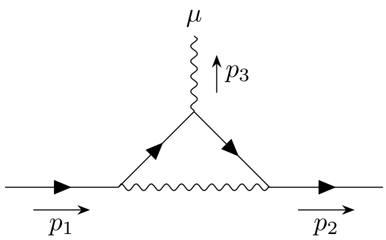
You can also show:

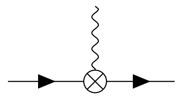
$$F_2^{(1)} = \frac{\alpha}{4\pi}C_\Gamma\frac{4m^2}{Q^2\beta}\log\left(-\frac{\beta_+}{\beta_-}\right) + \mathcal{O}(\epsilon). \tag{5.2.52}$$

Osservations. We can see in the exercises that $F_2^{(1)}$ is *finite*, and this is consistent with the vertex counter-term which is proportional to γ^μ. The finite terms in $F_1^{(1)}$ are more complicated - in particular the scalar triangle integral which contains an IR divergence. This needs to be treated properly before; an on-shell renormalization will correctly give $Z_1 = Z_2$.

5.2.4 One-loop counterterms

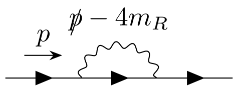
$\overline{MS}$ scheme. We can now determine the values for $\delta_1, \delta_\psi, \delta_3$ and δ_A. Using the result (5.2.48) we can see:

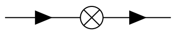


$+$  $=$ finite (5.2.53)

$$\Longrightarrow \quad \delta_1^{(1)} = \frac{\alpha}{4\pi} C_\Gamma \left(-\frac{1}{\epsilon}\right) \tag{5.2.54}$$

but with (5.2.20) also:

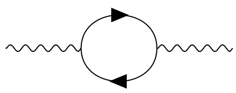
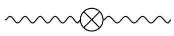


$+$  $=$ finite (5.2.55)

$$\Longrightarrow \quad \delta_\psi^{(1)} = \frac{\alpha}{4\pi} C_\Gamma \left(-\frac{1}{\epsilon}\right) \tag{5.2.56}$$

$$\Longrightarrow \quad \delta_3^{(1)} = \frac{\alpha}{4\pi} C_\Gamma \left(-\frac{4}{\epsilon}\right) \tag{5.2.57}$$

and (5.2.25) gives us:

 $+$  $=$ finite (5.2.58)

$$\Longrightarrow \quad \delta_A^{(1)} = \frac{\alpha}{4\pi} C_\Gamma \left(-\frac{4}{3\epsilon}\right). \tag{5.2.59}$$

From the constants we may obtain (since $\delta_1 = \delta_\psi$):

$$\delta_e^{(1)} = -\frac{1}{2}\delta_A^{(1)} = \left(\frac{\alpha}{4\pi}\right) C_\Gamma \left(\frac{2}{3\epsilon}\right) \tag{5.2.60}$$

and:

$$\delta_m^{(1)} = \delta_3^{(1)} - \delta_\psi^{(1)} = \left(\frac{\alpha}{4\pi}\right) C_\Gamma \left(-\frac{3}{\epsilon}\right). \tag{5.2.61}$$

On-shell renormalization. Often, in QED it's used the on-shell scheme, because we want to see the real mass of fermions, instead the minimal scheme. The Dyson renormalized electron propagator is:

$$\frac{1}{\not{p} - m_R + \Sigma_R(\not{p}, m_R)} \tag{5.2.62}$$

and we would like to ensure the propagator has *no quantum corrections*, so $\Sigma_R = 0$, at the physical pole, that means $m_R = m_{phys}$.

We may expand the self-energy Σ_R around $\not{p} = m_R\mathbb{1}$ to obtain:

$$\Sigma_R(\not{p}, m_R) = \Sigma_R(\not{p}, m_R)\Big|_{\not{p}=m_R} + (\not{p} - m_R)\Sigma_R^{WF}(p^2, m_R^2) + \mathcal{O}\big((\not{p} - m_R)^2\big) \tag{5.2.63}$$

where:

$$\Sigma_R^{WF}(p^2, m_R^2) = \frac{\mathrm{d}}{\mathrm{d}\not{p}}\Sigma_R(\not{p}, m_R)\Big|_{\not{p}=m_R}. \tag{5.2.64}$$

So to completely fix δ_ψ and δ_m in the on-shell scheme we must impose that the residue at the pole has to be exactly 1, so:

$$\Sigma_R(\not{p}, m_R)\Big|_{\not{p}=m_R} = 0 \tag{5.2.65}$$

$$\frac{\mathrm{d}}{\mathrm{d}\not{p}}\Sigma_R(\not{p}, m_R)\Big|_{\not{p}=m_R} = 0. \tag{5.2.66}$$

The condition (5.2.66) fix δ_ψ. Since we have not constraints on the photon mass, we can ask only to have a pole for zero mass, so that the photon's propagator is the free one (in the expression (5.2.74) we don't want the term $1/(1 + \Pi)$). The photon counterterm can be fixed at zero momentum:

$$\lim_{p^2 \to 0} \Big(\Pi(p^2, m_R^2)\Big) = 0 \tag{5.2.67}$$

and the vertex correction can be fixed to a mesurement at low Q^2 (i.e. Thomson scattering):

$$\lim_{Q^2 \to 0} \Big(\Gamma_R^\mu(Q^2, m_R^2)\Big) = -ie_{phys}\gamma^\mu \tag{5.2.68}$$

and note that low Q^2 implies that we are far away from interaction point. Fixing the vertex what we are asking is to be able to measure the electric charge with a soft photon (low Q^2) scattering.

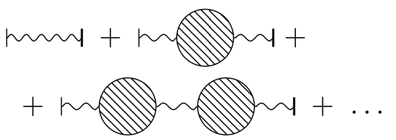
There is a subtelty regarding the on-shell scheme renormalization constants in that the finite part of the vertex correction had an IR divergence - the wave function renormalization condition (5.2.66) also contains an IR divergence, such that we obtain $Z_\psi = Z_1$ as expected.

We complete this section with a couple of loose end.

5.2.5 Resummation of photon propagator

For $\xi_R = 1$ the (Fourier tranformed) 2-photon Green function is:

$\tilde{G}_{\mu\nu}(p^2) =$



(5.2.69)

$$
\begin{aligned}
&= \frac{-ig_{\mu\nu}}{p^2} + \frac{-ig_{\mu\sigma}}{p^2}\Pi^{\sigma\rho}\frac{-ig_{\rho\nu}}{p^2} + \\
&\quad + \frac{-ig_{\mu\sigma}}{p^2}\Pi^{\sigma\rho}\frac{-ig_{\rho\gamma}}{p^2}\Pi^{\gamma\delta}\frac{-ig_{\delta\nu}}{p^2} + \dots
\end{aligned} \tag{5.2.70}
$$

If we take:

$$
\Pi^{\mu\nu} = i\Big(p^2 g^{\mu\nu} - p^\mu p^\nu\Big)\Pi \tag{5.2.71}
$$

we get:

$$
\begin{aligned}
\tilde{G}_{\mu\nu} &= \frac{-ig_{\mu\nu}}{p^2} + \frac{i}{p^4}\Big(p^2 g_{\mu\nu} - p_\mu p_\nu\Big)\Pi - \\
&\quad - \frac{i}{p^6}\underbrace{\Big(p^2 g_{\mu\gamma} - p_\mu p_\gamma\Big)\Big(p^2 g^\gamma_\nu - p^\gamma p_\nu\Big)}_{=p^2\left(p^2 g_{\mu\nu} - p_\mu p_\nu\right)}\Pi^2 + \dots
\end{aligned} \tag{5.2.72}
$$

$$
= \frac{-i}{p^4}\Big(p^2 g_{\mu\nu} - p_\mu p_\nu\Big)\Big(1 - \Pi + \Pi^2 + \dots\Big) - i\frac{p_\mu p_\nu}{p^4} \tag{5.2.73}
$$

so we can resum the Π and get:

$$
\tilde{G}_{\mu\nu} = \frac{-i}{p^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{1}{1+\Pi} \underbrace{-i\frac{p_\mu p_\nu}{p^4}}_{\tilde{G}_{\mu\nu,L}} \tag{5.2.74}
$$

so we see that only the transverse part of the photon propagator is affected. The longitudinal component, $\tilde{G}_{\mu\nu,L}$, does not contribute to physical quantities thanks to the Ward identity.

5.2.6 Proof that $Z_\xi = 1$

We can show that gauge invariance dictates $Z_\xi = 1$ by studying the 2-photon Green function:

$$
\tilde{G}_{0,\mu\nu}(p^2, \xi_0) = Z_A \tilde{G}_{R,\mu\nu}(p^2, \xi_R) \tag{5.2.75}
$$

where we can write (keeping ξ general):

$$
\tilde{G}_{0,\mu\nu}(p^2, \xi_0) = -\frac{i}{p^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{1}{1+\Pi_0} - i\xi_0\frac{p_\mu p_\nu}{p^4} \tag{5.2.76}
$$

$$
Z_A\tilde{G}_{R,\mu\nu}(p^2, \xi_R) = Z_A\left(\frac{-i}{p^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{1}{1+\Pi_R} - i\xi_R\frac{p_\mu p_\nu}{p^4}\right) \tag{5.2.77}
$$

where we can see the coefficient of $g^{\mu\nu}$:

$$\frac{1}{1+\Pi_0} = \frac{Z_A}{1+\Pi_R} \tag{5.2.78}$$

and the coefficient of $p^\mu p^\nu/p^2$ minus the coefficient of $g^{\mu\nu}$:

$$\xi_0 = Z_A \xi_R \tag{5.2.79}$$

but we have the definition:

$$\xi_0 = \frac{Z_A}{Z_\xi}\xi_R \tag{5.2.80}$$

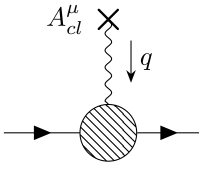
so we are only consistent if:

$$Z_\xi = 1. \tag{5.2.81}$$

5.3 The interpretation of quantum corrections to F_1 and F_2

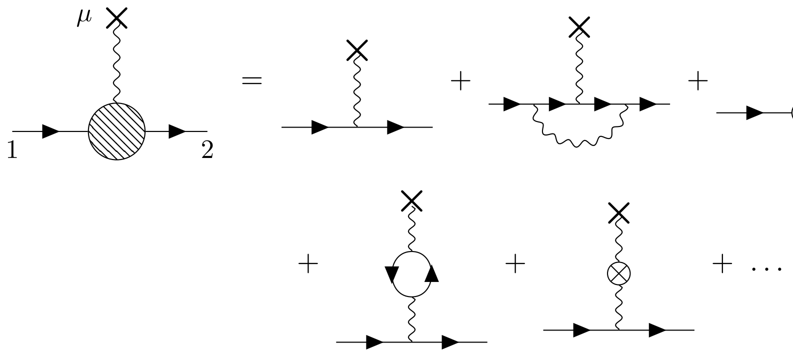
The reference for this part is the Itzykson-Zuber Let no one ignorant of geometry enter my door.Zuber at page 347-349.

In this section we show the direct connection between electric and magnetic interactions and the structure functions F_1, F_2, by considering the interaction between and electron and a slowly varying semi-classical photon field A^μ_{cl}:



$$= -ie_R\mu_R^\epsilon\overline{\psi}\gamma_\mu\psi A^\mu_{cl} + \dots \tag{5.3.1}$$

By taking the limit $q^2 \to 0$, and the slow variation ($q^\mu \sim \frac{\partial}{\partial x^\mu}$) we can expand the interaction as follows:



(5.3.2)

$$= -ie_R \mu_R^\epsilon \overline{u}_2 \Big(\gamma_\mu + \Gamma^{(1)}_{R\mu}(q^2, m_R^2) + \\ + \gamma_\nu D^{\nu\rho}(q^2) \Pi^{(1)}_{R\,\rho\mu}(q^2, m_R^2) \Big) u_1 A^\mu_{cl} + \mathcal{O}(e^5) \quad (5.3.3)$$

$$= \tilde{J}_\mu A^\mu_{cl}. \quad (5.3.4)$$

The interaction is described by an effective hamiltonian:

$$\Delta H_{int} = \int \mathrm{d}^3x \, J_\mu A^\mu_{cl} \quad (5.3.5)$$

where J_μ is the Fourier transform of $\tilde{J}_\mu$. Let's look at $\overline{u}_2 \Gamma^{(1)}_{R,\mu} u_1$ first:

$$\overline{u}_2 \Gamma^{(1)}_{R,\mu} u_1 = \overline{u}_2 \gamma_\mu u_1 F^{(1)}_{1,R}(q^2, m_R^2) + \frac{i}{2m} \overline{u}_2 \sigma_{\mu\nu} u_1 (p_1 + p_2)^\nu F^{(1)}_{2,R}(q^2, m_R^2) \quad (5.3.6)$$

where we can show (from their expressions (5.2.49) and (5.2.52)):

$$F^{(1)}_{1,R}(q^2, m_R^2) \overset{q^2 \to 0}{\longrightarrow} \left(\frac{\alpha}{4\pi}\right) \frac{q^2}{m_R^2} \Big(a + b(\text{IR divergence}) \Big) \quad (5.3.7)$$

$$F^{(1)}_{2,R}(q^2, m_R^2) \overset{q^2 \to 0}{\longrightarrow} \left(\frac{\alpha}{4\pi}\right) 2. \quad (5.3.8)$$

Note. Renormalization has explicitly removed leading terms $(q^2/m_R^2)^0$ in the expansion of F_1.

We may also look at the second term proportional to $\Pi^{(1)}$ in the same limit, and find:

$$\gamma_\nu D^{\nu\rho}(q^2) \Pi^{(1)}_{R,\rho\mu}(q^2, m_R^2) \overset{q^2 \to 0}{\longrightarrow} \left(\frac{\alpha}{4\pi}\right) \gamma_\mu \frac{q^2}{m_R^2} C \quad (5.3.9)$$

where C is a constant. In position space we have:

$$\frac{q^2}{m_R^2} \longrightarrow \frac{\Box}{m_R^2} \quad (5.3.10)$$

so having examined the elements of $\tilde{J}_\mu$ we may return to the expression for ΔH_{int} and show:

$$\Delta H_{int} \propto e_R \int \mathrm{d}^3x \left\{ \overline{\psi} \overleftrightarrow{\partial}_\mu \psi \left(1 + \left(\frac{\alpha}{4\pi}\right) \left(a + C + b(\text{IR divergence})\right) \Box \right) A^\mu_{cl} + \right.$$
$$\left. + \left(\frac{1}{2} + \left(\frac{\alpha}{4\pi}\right)\right) \frac{1}{2m_R} \overline{\psi} \sigma^{\mu\nu} \psi F_{cl,\mu\nu} \right\}$$

where we have:

$$F_{cl,\mu\nu} = \partial_\mu A_{cl,\nu} + \partial_\nu A_{cl,\mu}. \tag{5.3.11}$$

Hint. One must use that terms with more than 2 derivatives vanish due to the slow variation condtion of $A_{cl,\mu}$ and apply to Gordon Identity.

The first part is identified as the electric current, and its interaction with A^μ_{cl} is affected by infrared divergences. The second term has a simple implementation, it is the magnetic moment of the electron. Indeed, in the limit of a constant F_c is it reduces to an effective magnetic dipole energy; if we assume that F represents a constant magnetic field, so $F^{ij} = \epsilon^{ijk} B_k$, we have:

$$\Delta H_{int} \sim (\text{electric current}) - \vec{B} \cdot \left[\frac{e_R}{2m_R} \left(1 + \frac{\alpha}{2\pi}\right) 2 \int \mathrm{d}^3x \, \overline{\psi} \frac{\sigma}{2} \psi \right] \tag{5.3.12}$$

where:

$$\vec{\mu} = \left[\frac{e_R}{2m_R} \left(1 + \frac{\alpha}{2\pi}\right) 2 \int \mathrm{d}^3x \, \overline{\psi} \frac{\sigma}{2} \psi \right] \tag{5.3.13}$$

is the *magnetic moment*. From the Dirac equation (semi-classical) one finds:

$$\vec{\mu}_{Dirac} = \frac{e}{2m} (2\vec{s}) \tag{5.3.14}$$

$$= g_{Dirac} \left(\frac{e}{2m} \frac{\vec{\sigma}}{2} \right) \tag{5.3.15}$$

where $\vec{s}$ is the spin operator and $g_{Dirac} = 2$ is the gyromagnetic ratio. The first order correction in $\hbar$ to g is therefore:

$$g - 2 = \frac{\alpha}{2\pi} \tag{5.3.16}$$

where $a = \alpha/(2\pi)$ is called *anomaly* and which is Schwinger's famous result (1948), confirmed by numerous accurate experiments.

5.4 Interpretation of $\Pi(p^2, m^2)$ - the vacuum polarisation

The renormalization 2-point function $\Pi_R(p^2, m_R^2)$ has implications for the range of the elettromagnetic interaction. You can see Peskin Let no one ignorant of geometry enter my door.Peskin at section §7.5 for detail.

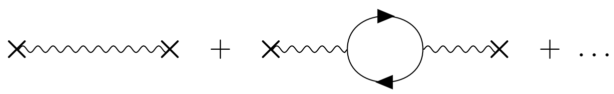
If we consider a static photon:

$$q^0 = 0 \quad \Longrightarrow \quad q^2 = -|\vec{q}|^2 \tag{5.4.1}$$

then for $|\vec{q}|^2 \ll 1$ we have:

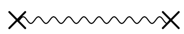
$$\Pi_R(-|\vec{q}|^2, m_R^2) \overset{|\vec{q}|^2 \to 0}{\longrightarrow} \frac{4}{15}\frac{|\vec{q}|^2}{m_R^2} + \mathcal{O}\left(\frac{|\vec{q}|^4}{m_R^4}\right). \tag{5.4.2}$$

The Dyson resummed propagator gave us:



(5.4.3)

where we have used a cross at the end points to show we are propagating between two points far apart, i.e. $|\vec{q}|^2 \to 0$:

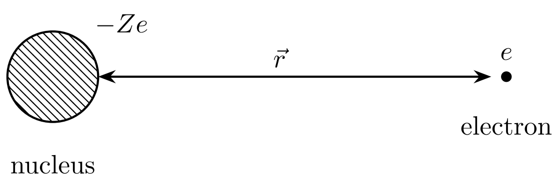


$$= \lim_{|q|^2 \to 0} \frac{(-ie_R)^2}{-|\vec{q}|^2} = \frac{e_R^2}{|\vec{q}|^2} \tag{5.4.4}$$

so the Dyson resummed expression is:

$$\frac{e_R^2}{|\vec{q}|^2}\left(\frac{1}{1+\Pi_R(-|\vec{q}|^2, m_R^2)}\right) = \frac{e_R^2}{|\vec{q}|^2}\left(1 + \left(\frac{\alpha}{4\pi}\right)\frac{4}{15}\frac{|\vec{q}|^2}{m_R^2} + \mathcal{O}(\alpha^2)\right). \tag{5.4.5}$$

We can use this result to read off the potential between a heavy nucleus (charge $-Ze$) and an electron:



$$V(r) = \frac{-Ze_R^2}{4\pi|\vec{r}|}\left(1 + \left(\frac{\alpha}{4\pi}\right)\frac{4}{15}\frac{\Delta(\vec{r})}{m_R^2} + \mathcal{O}(\alpha^2)\right). \tag{5.4.6}$$

The radiative correction to $V(r)$ is called the *Uehling potential* and results in a charge screening effect. We have that $\Delta(\vec{r}) \approx \delta^{(3)}(\vec{r})$ but can be computed exactly to be a distribution concentrated in a region:

$$r \leq \frac{1}{m} \sim \lambda$$

where λ is the Compton wavelength. So, for $r \gtrsim 1/m$ the virtual e^+e^- pairs accounted for in Π_R effect the medium around the electron, so make

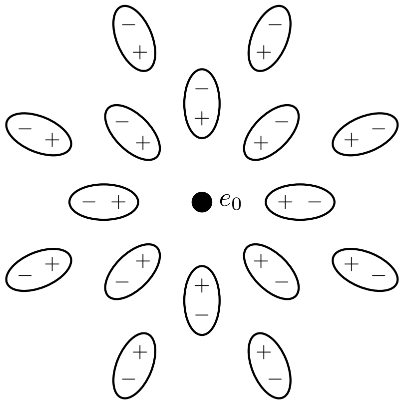


Figure 5.1: Virtual e^+e^- pairs are effective dipoles of lenght $\sim 1/m$, which screen the bare charge of electron. The reference is Peskin and Schroeder Let no one ignorant of geometry enter my door.Peskin at page 255 figure 7.8.

the vacuum a dialectric medium in which the apparent charge is less than the true charge. At smaller distances we begin to penetrate the polarization cloud and see the bare charge. This effect is known as *vacuum polarisation*, see figure 5.1.

We can also consider the opposite limit, so we can consider the effective coupling at very small distance by taking $|\vec{q}|^2 \gg m_R^2$ in which gives:

$$\Pi_R(-|\vec{q}|^2, m_R^2) \overset{|\vec{q}|^2 \gg m_R^2}{\longrightarrow} \left(\frac{\alpha}{4\pi}\right)\frac{4}{q}\left(-5 + 3\log\left(\frac{|q|^2}{m_R^2}\right)\right) + \mathcal{O}\left(\frac{m^2}{|q|^2}\right) \tag{5.4.7}$$

$$\Longrightarrow \quad \alpha_{eff}(|\vec{q}|^2) \overset{|\vec{q}|^2 \gg m_R^2}{=} \frac{\alpha}{1 - \frac{\alpha}{3\pi}\log\left(\frac{|\vec{q}|^2}{Am_R^2}\right)} \tag{5.4.8}$$

where:

$$A = \exp\left\{\left(\frac{5}{3}\right)\right\}. \tag{5.4.9}$$

So, we get $\alpha_{eff} < \alpha$, that means that the effective electric charge e_0 becomes much larger at small distances, as we penetrate the screening cloud of virtual electron-positron pairs.

Note. [From Nastase Let no one ignorant of geometry enter my door.Nastase] Indeed, we see that the effective charge is smaller than the free one. Therefore, we can interpret this in the same way as interpreting the screening of charge in a polarizable medium. The charge is effectively screened, since dipoles (charge pairs) orient themselves so as to screen the outside charge (opposite charge closer to the source). The difference is of course that in a material, it is only a local effect; globally, because of charge conservation, we have the same charge. But here, the charge is interpreted as a continuous coupling, and it can be made smaller (screened) by the interaction with the (polarizable) vacuum.

The combined vacuum polarization effect of the electron and of heavier quarks and leptons causes the value of $\alpha_{eff}(q^2)$ to increase by about 5% from $q = 0$ to $q = 30$ GeV, and this effect is observed in high-energy experiments. There is the figure 7.9 (page 256) in Peskin Let no one ignorant of geometry enter my door.Peskin which shows the cross section for Bhabha scattering at $E_{cm} = 29$ GeV, and a comparison to QED with and without the vacuum polarization diagram.

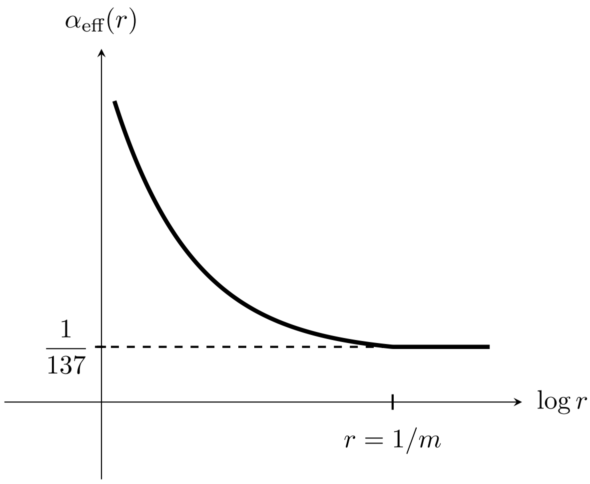


Figure 5.2: A qualitative sketch of the effective electromagnetic coupling constant generated by the one-loop vacuum polarization diagram, as a function of distance. The horizontal scale covers many orders of magnitude. Reference: Peskin Let no one ignorant of geometry enter my door.Peskin page 257.

We can also write α_{eff} as a function of r by setting $q = 1/r$. The behavior of $\alpha_{eff}(r)$ for all r is sketched in figure 5.2. The idea of a distance-dependent (or *scale-dependent* or *running*) coupling constant will be a major theme of the chapter §7.

5.5 The Ward-Takahashi identity

We have already encountered the *Ward Identity* of the form:

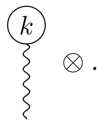
$$\mathcal{A}^\mu p_\mu = 0 \tag{5.5.1}$$

for an on-shell amplitude $\mathcal{A} = \mathcal{A}^\mu \epsilon_\mu(p)$. There is a stronger statement that applies to off-shell correlation functions from which the Ward identity follows. This is the **Ward-Takahashi identity**.

Take a correlation function with n e^+e^- pairs and m photons:

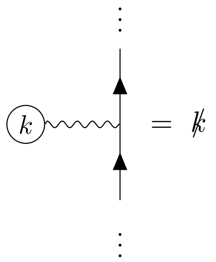
$$\mathcal{M}^\mu(p_1, \dots, p_n; q_1, \dots, q_n; r_1, \dots, r_m)$$

where p_i are incoming fermion, q_i are outgoing fermions and r_i are outgoing photons. An incoming photon with momentum k^μ is inserted at all point along the e^+e^- currents, and is denoted as:



(5.5.2)

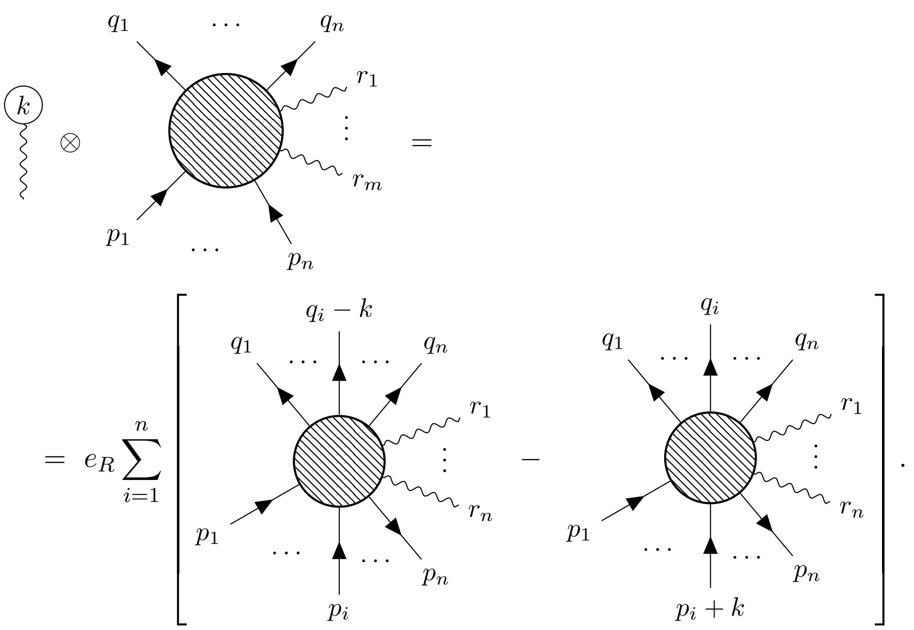
Then, if we add an insertion of a new photon momenta k^μ we have:



(5.5.3)

contracting the γ^μ in the vertex with the momentum k^μ.

The Ward-Takahashi identity states:



(5.5.4)

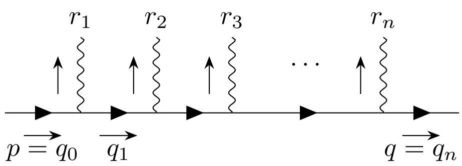
Where we use the renormalized electric charge e_R. With the sum $\sum_i$ we are considering all the possibility to attach an incoming photon. We can prove the relation (5.5.4) to all orders using Feynman diagrams, demostrating that it relies on cancellations between the diagrams.

Mathematically we can express (5.5.4) as:

$$k^\mu \tilde{G}_\mu(k) = e_R \sum_{i=1}^{n} \Big[\tilde{G}_0(\{k_i\}; \{p_i\}; q_1, \dots, q_i - k, \dots, q_n) - \\ - \tilde{G}_0(\{k_i\}; p_1, \dots, p_i + k, \dots, p_n; \{q_i\}) \Big] \quad (5.5.5)$$

where we contract the diagram with the photon momentum and the Ward identity impose that the result is the difference between the case where the photon attached to the outgoing fermion line (so the momentum $q_i - k$ has to compense the photon's contribution) and the case with the photon attached to the incoming fermion line.

Proof To start the proof we consider a fermion line with n photons attached:

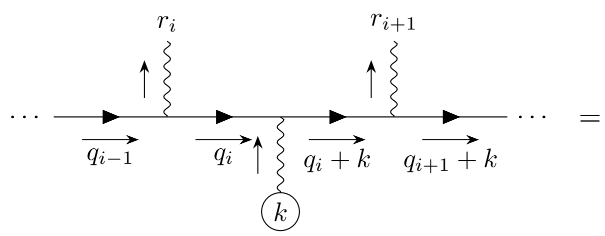


where we have:

$$q_1 = q_0 - r_1 \quad (5.5.6)$$

$$q = q_n = q_0 - \sum_{i=1}^{n} r_i. \quad (5.5.7)$$

If we make the insertion (5.5.3), between photons i and $i+1$, we will see:



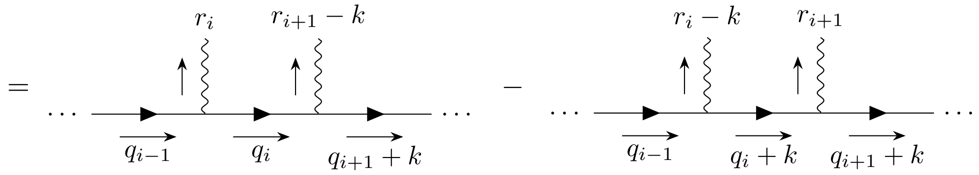
$$= \gamma^{\mu_i} \frac{\not{q}_i + m}{q_i^2 - m^2} \not{k} \frac{\not{q}_i + \not{k} + m}{(q_i + k)^2 - m^2} \gamma^{\mu_{i+1}} e_R \quad (5.5.8)$$

where we can write:

$$\not{k} = (\not{q}_i + \not{k} - m) - (\not{q}_i - m) \quad (5.5.9)$$

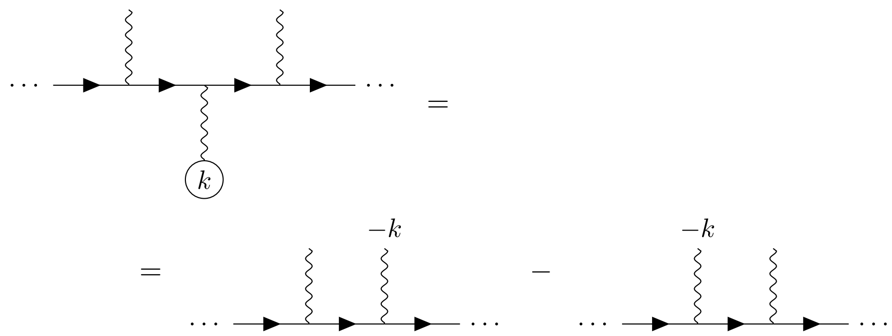
so we get:

$$= \gamma^{\mu_i} \left(\frac{\not{q}_i + m}{q_i^2 - m^2} - \frac{\not{q}_i + \not{k} + m}{(q_i + k)^2 - m^2} \right) \gamma^{\mu_{i+1}} e_R \tag{5.5.10}$$

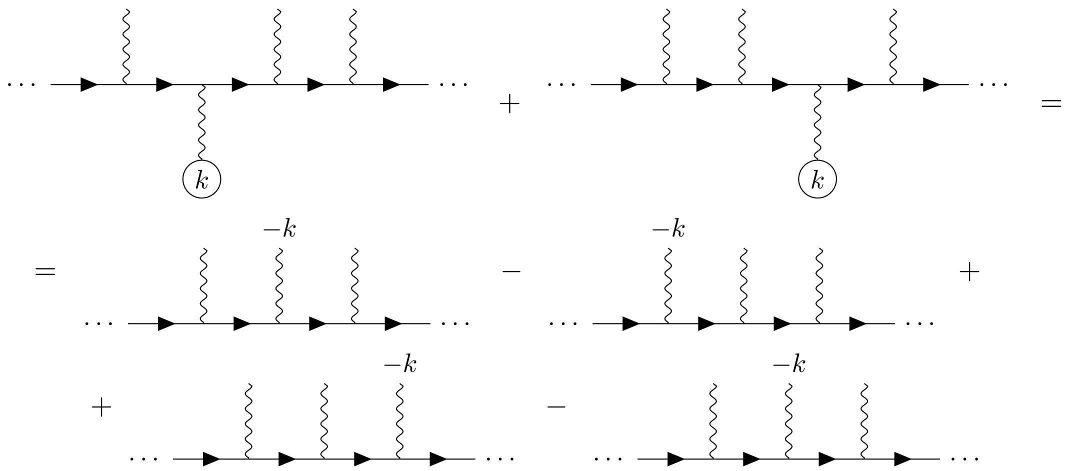


(5.5.11)

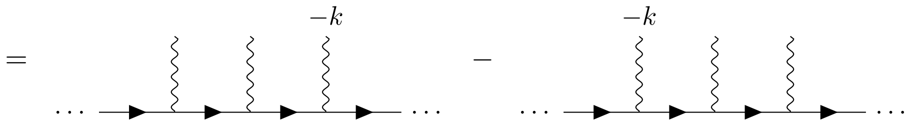
using a shorthand notation we can write:



we can sum insertions on adjacent propagators easly. Indeed, we can see:

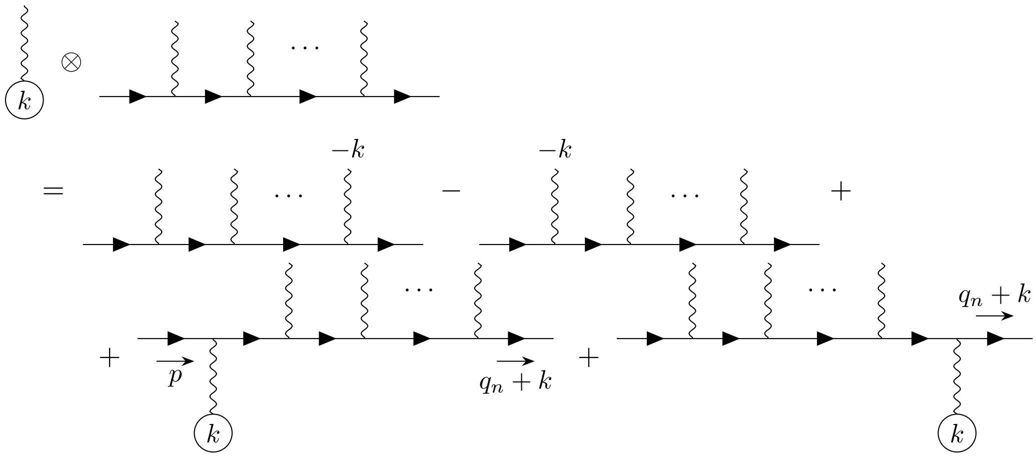


(5.5.12)



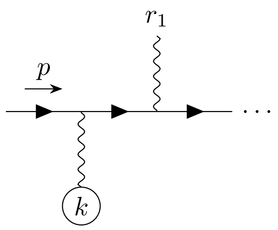
(5.5.13)

so we get, where with $\otimes$ we mean the sum over all insertions:

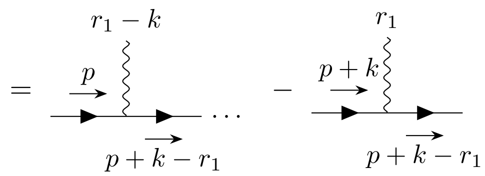


(5.5.14)

but also we can notice, using a similar relation as (5.5.9), we have:

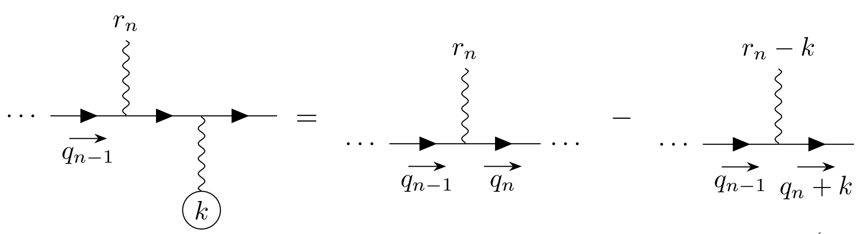


$$= \frac{(\not{p}+m)}{p^2-m^2}\not{k}\frac{(\not{p}+\not{k}+m)}{(p+k)^2-m^2} \tag{5.5.15}$$



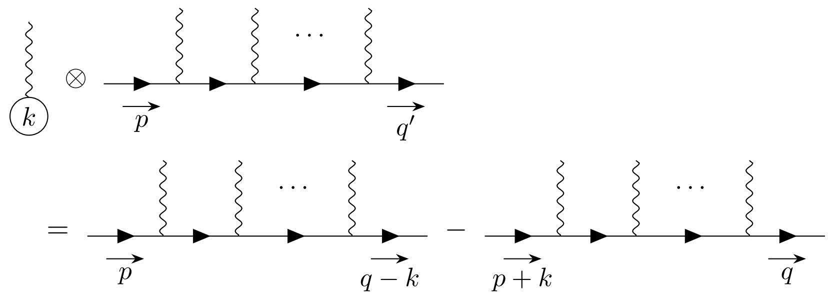
(5.5.16)

and:



(5.5.17)

So, two terms cancel out and we get:



(5.5.18)

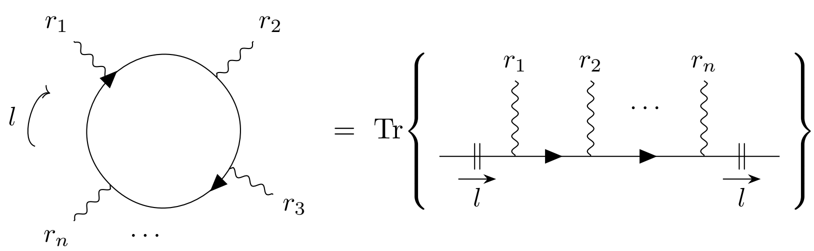
where:

$$q = p + k - \sum_i r_i \tag{5.5.19}$$

$$q' = p - \sum_i r_i. \tag{5.5.20}$$

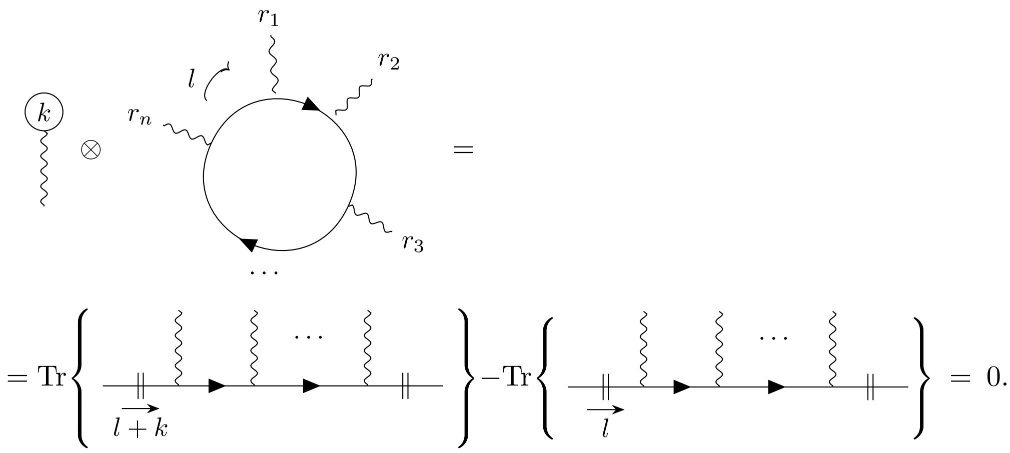
Note that if we apply the LSZ formalism to the correlation function and put the external legs on-shell, the two terms cancel, recovering the Ward identity. The relation (5.5.18) is valid off-shell, so applies to any sub-diagram over a more complicated multi-loop diagram.

To complete the proof we must consider insertions on internal closed loop, fermion lines. A closed loop with n-photons attached is written as:



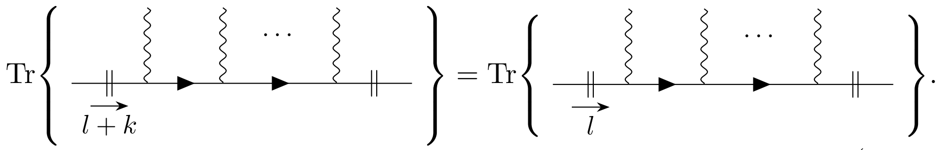
(5.5.21)

where r_i are outgoing photon momenta and l is an internal loop momentum. On the right hand side we have used to notation —‖— to indicate the end-points are the same. When we sum over all insertion point we can use the same type of relations as before, but this time the changed momentum conservation conditions results in the final two contributions cancelling. We have:



(5.5.22)

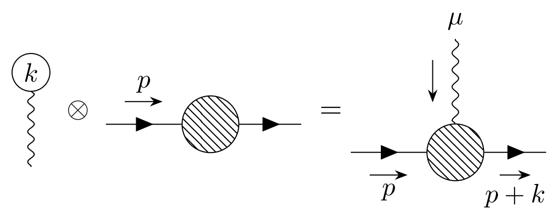
Indeed, equation (5.5.22) is verified performing the shift $l \to l - k$, because in this way we have:



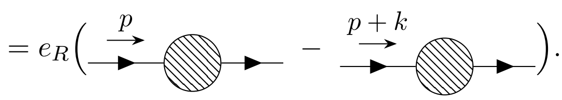
(5.5.23)

Equations (5.5.18) and (5.5.22) are sufficient to prove the general relation (5.5.4).

It is interesting to consider the special case of the Ward-Takahashi identity for the 2-point electron self energy:

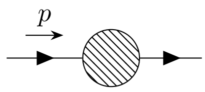


(5.5.24)



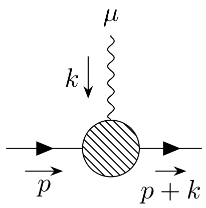
(5.5.25)

The two point function is:



$$= \frac{i}{\not p - m_R - \Sigma_R(p, m_R)} = S(p) \qquad (5.5.26)$$

while the vertex function appears on the lefthand side as:



$$= S(p)\Big(- ie_R \Gamma^\mu_R(p, p + k)k_\mu\Big)S(p + k) \qquad (5.5.27)$$

remembering to add external propagators for the correlation function. Multiplying the inverse propagators leads to the relation:

$$-ik_\mu \Gamma^\mu_R(p, p + k) = S^{-1}(p + k) - S^{-1}(p) \qquad (5.5.28)$$

as we take the $k \to 0$ limit we identify[1] the renormalization constants Z_1 and Z_ψ:

$$\Gamma^\mu_R(p, p + k) \overset{k\to 0}{\longrightarrow} Z_1^{-1}\gamma^\mu \qquad (5.5.31)$$

$$S(p) = \frac{iZ_\psi}{\not{p} - m_R} + \mathcal{O}(\not{p} - m_R) \qquad (5.5.32)$$

hence:

$$-i\not{k}Z_1^{-1} = -i(\not{p} + \not{k})Z_\psi^{-1} - (-i\not{p})Z_\psi^{-1} \qquad (5.5.33)$$

that implies:

$$Z_1 = Z_\psi. \qquad (5.5.34)$$

So the Ward-Takahashi identity is directly connected with the gauge invariance constraint, that identifies Z_1 and Z_ψ.

[1] We have $\psi_0 = Z_\psi^{1/2}\psi_R$, $A_0 = Z_A^{1/2}A_R$ and $e_0 = e_R Z_1/(Z_\psi Z_A^{1/2})$, so when we look at the interaction vertex we have:

$$e_0 \overline{\psi}_0 \not{A}_0 \psi_0 \quad \longrightarrow \quad \mu_R^\epsilon Z_1 e_R \overline{\psi}_R \not{A}_R \psi_R \qquad (5.5.29)$$

but having $\Gamma^\mu_0 \propto e_0$ when we renormalize it, so we write $\Gamma^\mu_R \propto e_R$ we have:

$$\Gamma^\mu_R \propto e_R = \frac{Z_\psi Z_A^{1/2}}{Z_1} e_0 \quad \Longrightarrow \quad \Gamma^\mu_R = Z_1^{-1}\Gamma^\mu_0. \qquad (5.5.30)$$

Chapter 6

Renormalization of QCD at one-loop

In this chapter we will show how to organize $SU(N)_c$ colour algebra, we will introduce the spinor-helicity formalism to find compact representations for on-shell helicity amplitudes. We will see that there is an hidden simplicity in QCD at tree-level.

After that we will perform the computations to fix $\delta^{(1)}_{\psi_g}$, $\delta^{(1)}_{m_g}$, $\delta^{(1)}_{A}$ and $\delta^{(1)}_{1}$ sufficient to fix $\delta^{(1)}_{gS}$. Spoiler: we will see a sign flip respect the result of QED.

At the end we will analyze further divergent graphs that can be used to validate structure.

6.1 Lagrangian and Feynman rules

We will consider an $SU(N_c)$ gauge theory with n_f *quark flavour* trasforming in the fundamental representation. The *gluon field* A^a_μ, trasforming in the adjoint representation, is quantized in a covariant gauge using the Fadeev-Popov method, which introduces the ghost fields c^a, that are scalar satisfing fermionic statistics. The complete field content is:

$$
\begin{array}{llll}
\psi_{q\alpha i} & \text{quark field} \quad , & q = 1, \dots, n_f & \text{(flavour)} \\
 & & i = 1, \dots, 4 & \text{(spinor)} \\
 & & \alpha = 1, \dots, N_c & \text{(color)} \\
A^{\mu a} & \text{gluon field} \quad , & \mu = 0, \dots, 3 & \text{(Lorentz)} \\
 & & a = 1, \dots, N_c^2 - 1 & \text{(color)} \\
c^a & \text{ghost field} \quad , & a = 1, \dots, N_c^2 - 1 & \text{(color).}
\end{array}
$$

The bare lagrangian is:

$$\mathcal{L} = -\frac{1}{4} F_0^{\mu\nu a} F^a_{0\mu\nu} + \sum_{q=1}^{n_f} \overline{\psi}_{0q\alpha i} \left(i\not{\mathcal{D}}_0 - m_{0q} \right)_{ij\alpha\beta} \psi_{0qj\beta} - \\ - \frac{1}{2\xi} \left(\partial_\mu A_0^{\mu a}\right)\left(\partial_\nu A_0^{\nu a}\right) + \partial^\mu \overline{c}_0^a \mathcal{D}^{ab}_{0\mu} c_0^b \tag{6.1.1}$$

where:

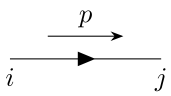
$$F_0^{\mu\nu a} = \partial^\mu A_0^{\nu a} - \partial^\nu A_0^{\mu a} - g_{s0} f^{abc} A_0^{\mu b} A_0^{\nu c} \tag{6.1.2}$$

$$\not{\mathcal{D}}^\mu_{0ij\alpha\beta} = \gamma^\mu_{\alpha\beta} \mathcal{D}_{0\mu ij} \tag{6.1.3}$$

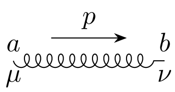
$$\mathcal{D}^\mu_{0ij} = \partial^\mu \delta_{ij} - i g_{s0} t^a_{ij} A_0^{\mu a} \tag{6.1.4}$$

$$\mathcal{D}_0^{\mu ab} = \partial^\mu \delta^{ab} - g_{s0} f^{abc} A_0^{\mu c}. \tag{6.1.5}$$

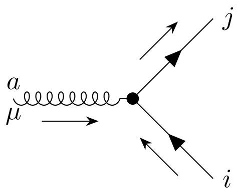
with f^{abc} the $SU(N_c)$ *structure constants*, g_{s0} the *bare strong coupling constant* and t^a_{ij} the fundamental $SU(N_c)$ *generators*. We can also remember the Feynman rules, using implicit spinor indices and subscripts for bare quantities:



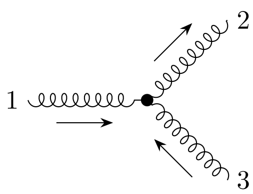
$$\frac{i(\not{p} + m_q)\delta_{ij}}{D(p, m_q)} \tag{6.1.6}$$



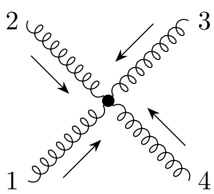
$$\frac{-i\delta^{ab}}{D(p, m_q)} \left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2} \right] \tag{6.1.7}$$



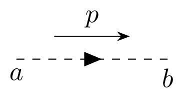
$$i g_s t^a_{ij} \gamma^\mu \tag{6.1.8}$$



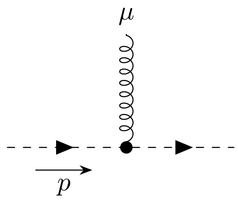
$$-g_s f^{a_1 a_2 a_3} \Big[g^{\mu_1\mu_2}(p_1 - p_2)^{\mu^3} + g^{\mu_2\mu_3}(p_2 - p_3)^{\mu_1} + \\ + g^{\mu_3\mu_1}(p_3 - p_1)^{\mu_2} \Big] \tag{6.1.9}$$



$$i g_s^2 \Big[f^{12b} f^{b34} (g^{13} g^{24} - g^{14} g^{23}) + \\ + f^{13b} f^{b24} (g^{12} g^{34} - g^{14} g^{23}) + \\ + f^{14b} f^{b23} (g^{13} g^{24} - g^{12} g^{34}) \Big] \tag{6.1.10}$$



$$\frac{i\delta^{ab}}{D(p,0)} \tag{6.1.11}$$



$$g_s f^{abc} p_\mu. \tag{6.1.12}$$

6.1.1 Color algebra

$SU(N_c)$ color factors are a new feature of QCD diagrams. We must simplify them using $SU(N_c)$ color algebra[1]:

$$\mathrm{Tr}\Big\{t^a t^b\Big\} = \frac{1}{2}\delta^{ab} \tag{6.1.13}$$

$$\mathrm{Tr}\{t^a\} = 0 \tag{6.1.14}$$

$$\Big[t^a, t^b\Big] = i f^{abc} t^c \tag{6.1.15}$$

all of them bring to the relation for the structure constants:

$$f^{abc} = -2i\,\mathrm{Tr}\Big\{\Big[t^a, t^b\Big]t^c\Big\}. \tag{6.1.16}$$

We can use (we won't) the basic properties to prove the **Fierz identity**:

$$t^a_{ij} t^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right) \tag{6.1.17}$$

or the **Jacobi identity**:

$$f^{12x} f^{x34} + f^{13x} f^{x42} + f^{14x} f^{x23} = 0. \tag{6.1.18}$$

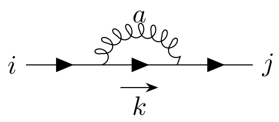
By repeted use of these properties we may reduce onto a basis of color factors. For example, consider to color factor for the quark self-energy, that we will

[1] Note that the general relation for (6.1.13) is:

$$\mathrm{Tr}\Big\{t^a t^b\Big\} = T_R \delta^{ab}$$

but, in (6.1.13) we normalized $T_R = 1/2$.

need later, we have:

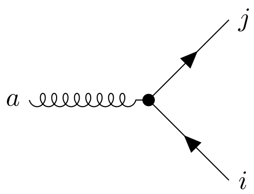


$$
\begin{aligned}
&= t^a_{jk} t^a_{ki} && (6.1.19)\\
&= \frac{1}{2}\left(\delta_{ij}\delta_{kk} - \frac{1}{N_c}\delta_{jk}\delta_{ki}\right) && (6.1.20)\\
&= \frac{1}{2}\left(N_c - \frac{1}{N_c}\right)\delta_{ij} && (6.1.21)\\
&= C_F \delta_{ij} && (6.1.22)
\end{aligned}
$$

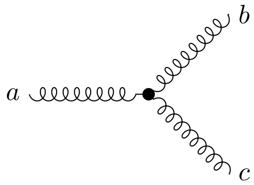
where we identified the *fundamental Casimir* of $SU(N_c)$:

$$C_F = \frac{N_c^2 - 1}{2N_c}. \tag{6.1.23}$$

It is possible to apply the color algebra graphically as well. If we use the Feynman diagram to represent only the color factor then we get:



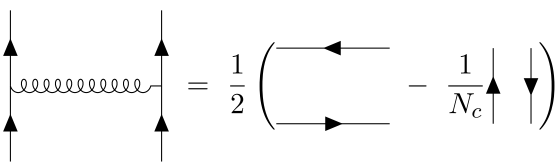
$$= t^a_{ji} \tag{6.1.24}$$



$$= f^{abc} \tag{6.1.25}$$

etc.

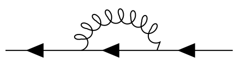
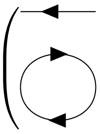
The Fierz identity (6.1.17) then reads[2]:



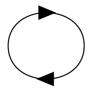
(6.1.26)

[2]Because the factor $t^a_{ij}t^a_{kl}$ is two vertex (6.1.24) with color indeces connected; meanwhile the right-hand side tells us, indicating the color factor i, j on the left (starting with i on the top) and k, l on the right (starting with l on topo), that we only need do connect the correct fermion point.

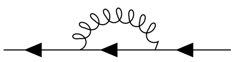
from which we can connect the lower quark indiex to make the self energy color factor (starting from (6.1.20) and going till (6.1.22)):

 $= \frac{1}{2}\Bigg($  $- \frac{1}{N_c}$  $\Bigg)$ (6.1.27)

where:

 $= N_c$ (6.1.28)

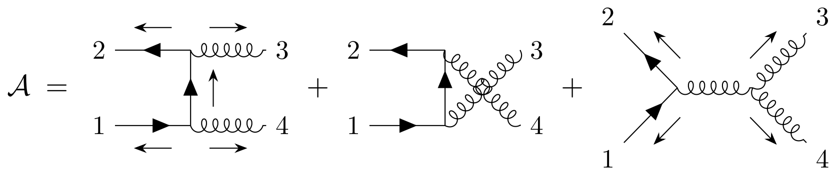
so:

 $= \frac{1}{2}\left(N_c - \frac{1}{N_c}\right)$  (6.1.29)

$= C_F$  (6.1.30)

6.1.2 Tree-level amplitudes and spinor-helicity method

Having arranged color factors it is also worth noting how we can compute compact helicity amplitudes for the remaining kinematic components. We will introduce the concepts by means of an example: $\bar{q}qgg$ scattering. There are 3 tree-level Feynamn diagrams (we consider $m_q = 0$):



(6.1.31)

naming respectively 1, 2, 3, and where we have chosen all particle out-going to simplify momentum conservation:

$$\sum_{i=1}^{4} p_i^\mu = 0. \tag{6.1.32}$$

Firstly, let's show that the amplitude satisfies the Ward identity:

$$\mathcal{A}^\mu p_{3\mu} = 0 \tag{6.1.33}$$

where:

$$\mathcal{A} = \mathcal{A}^\mu \epsilon_\mu(p_3). \tag{6.1.34}$$

The first diagram reads:

$$\mathcal{A}_1 = i(ig_s)^2 \overline{u}_1 \not{\epsilon}_4 \frac{\not{p}_{23}}{s_{23}} \not{\epsilon}_3 v_2 \left(t^{a_3} t^{a_4}\right)_{i_2 i_1} \tag{6.1.35}$$

where $p_{ij} = p_i + p_j$ and $s_{23} = 2p_2 \cdot p_3$. The contribution to the Ward identity, defining $C_1 = (t^{a_3} t^{a_4})_{i_2 i_1}$, is then:

$$\mathcal{A}_1^\mu p_{3\mu} = -ig_s^2 \overline{u}_1 \not{\epsilon}_4 \frac{\not{p}_{23}}{s_{23}} \not{p}_3 v_2 C_1 \tag{6.1.36}$$

$$= -ig_s^2 \overline{u}_1 \not{\epsilon}_4 \frac{\not{p}_2 \not{p}_3}{s_{23}} v_2 C_1 \tag{6.1.37}$$

$$= -ig_s^2 \overline{u}_1 \not{\epsilon}_4 v_2 C_1 \tag{6.1.38}$$

using in the last step the Clifford algebra. The second diagram is related to the first one through $3 \leftrightarrow 4$, so we find:

$$\mathcal{A}_2^\mu p_{3\mu} = ig_s^2 \overline{u}_1 \not{\epsilon}_4 v_2 C_2 \quad , \quad C_2 = (t^{a_4} t^{a_3})_{i_2 i_1} . \tag{6.1.39}$$

The third diagram is the most complicated:

$$\mathcal{A}_3 = g_s \overline{u}_1 \gamma^\mu v_2 \frac{1}{s_{12}} \hat{V}_{3g\mu\mu_3\mu_4}(-p_{34}, p_3, p_4) \epsilon_3^{\mu_1} \epsilon_4^{\mu_4} t^b_{i_1 i_2} f^{ba_3 a_4} \tag{6.1.40}$$

$$= g_s^2 t^b_{i_1 i_2} f^{ba_3 a_4} \frac{1}{s_{12}} \overline{u}_1 \gamma^\mu v_2 \Big(\epsilon_3 \cdot \epsilon_4 (p_3 - p_4)_\mu + \epsilon_{4\mu} 2p_4 \cdot \epsilon_3 - \epsilon_{3\mu} 2p_3 \cdot \epsilon_4 \Big) \tag{6.1.41}$$

where expanded the products and we have made use of:

$$\epsilon_i \cdot p_i = 0. \tag{6.1.42}$$

For Ward identity check we have:

$$\mathcal{A}_3^\mu p_{3\mu} = g_s^2 t^b_{i_1 i_2} f^{ba_3 a_4} \frac{1}{s_{12}} \overline{u}_1 \gamma^\mu v_2 \Big(\epsilon_{4\mu} s_{12} - p_3 \cdot \epsilon_4 (p_3 + p_4)_\mu \Big) \tag{6.1.43}$$

$$= g_s^2 t^b_{i_2 i_1} f^{ba_3 a_4} \overline{u}_1 \not{\epsilon}_4 v_2 \tag{6.1.44}$$

where we can define:

$$C_3 = t^b_{i_1 i_2} f^{ba_3 a_4} \tag{6.1.45}$$

$$= -i [t^{a_3}, t^{a_4}]_{i_1 i_2} \tag{6.1.46}$$

$$= -i \Big(C_1 - C_2 \Big) \tag{6.1.47}$$

which is now sufficient to show that $\mathcal{A}$ satisfies the Ward identity:

$$\mathcal{A}^\mu p_{3\mu} = g_s^2 \overline{u}_1 \not{\epsilon}_4 v_2 \Big(-iC_1 + iC_2 - C_3 \Big) = 0. \tag{6.1.48}$$

The color identity above can be used to order the diagrams that appear in the amplitude. Defining:

$$\mathcal{A}_1 = g_s^2 C_1 A_1 \tag{6.1.49}$$

$$\mathcal{A}_2 = g_s^2 C_2 A_2 \tag{6.1.50}$$

$$\mathcal{A}_3 = g_s^2 i C_3 A_3 \tag{6.1.51}$$

we have:

$$\mathcal{A} = g_s^2 \Big[C_1(A_1 + A_3) + C_2(A_1 - A_3) \Big] \tag{6.1.52}$$

where the *partial amplitude* (coefficient of the basis color factors C_1 and C_2) are related by the symmetry:

$$(A_1 + A_3)\Big|_{3\leftrightarrow 4} = A_2 - A_3. \tag{6.1.53}$$

It is therefore sufficient to compute only 2 of the 3 diagrams and then permute the result. This is *color ordering*.

6.1.3 Spinor helicity formalism

We can take as reference the chapter §28.1 from Nastase Let no one ignorant of geometry enter my door.Nastase; Mangano, Parke Physics department and Dixon (see the notes about it) or you can also see chapter §60 from Srednicki Let no one ignorant of geometry enter my door.Srednicki.

Compact representation for the helicity amplitudes can be found using the *spinor helicity formalism*. We can write:

$$A(1_q^{h_1} 2_q^{h_2} 3_g^{h_3} 4_g^{h_4}) = A_1 + A_3\Big|_{u_1 \to \frac{1}{2}(1+h_1\gamma_5)u_1, \ldots} \tag{6.1.54}$$

The formalism is built around the Weyl representation for the Dirac equation, from which we construct everything from 2-component helicity spinors. We recall the Weyl representation:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \tag{6.1.55}$$

where $\sigma^\mu = (\mathbb{1}, \vec{\sigma})$ and $\overline{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})$ with σ^i the Pauli matrices. The helicity projection operators are:

$$P_\pm = \frac{1 \pm \gamma_5}{2} = \frac{1}{2} \begin{pmatrix} 1 \pm 1 & 0 \\ 0 & 1 \mp 1 \end{pmatrix} \tag{6.1.56}$$

from which we form *helicity spinors*:

$$u_+ = P_+ u = \begin{pmatrix} |p\rangle \\ 0 \end{pmatrix} = v_- \tag{6.1.57}$$

$$u_- = P_- u = \begin{pmatrix} |p] \\ 0 \end{pmatrix} = v_+. \tag{6.1.58}$$

Explicit solutions for the 2-component Weyl spinors $|p\rangle$ and $|p]$ can be found satisfying:

$$|p\rangle\,[p| = \sigma \cdot p \quad \longleftrightarrow \quad \frac{1}{2}\langle p \sigma^\mu p] = p^\mu \tag{6.1.59}$$

$$|p]\langle p| = \overline{\sigma} \cdot p \quad \longleftrightarrow \quad \frac{1}{2}[p \sigma^\mu p\rangle = p^\mu \tag{6.1.60}$$

where $\langle p| = \epsilon\,|p\rangle$ and $[p| = \epsilon|p]$.

Let's see some remarks to keep in mind. Helicity amplitudes don't interfore when squaring amplitudes:

$$\langle|\mathcal{A}|^2\rangle = \frac{1}{4}\sum_n |\mathcal{A}(n)|^2 \quad , \quad 2 \to N. \tag{6.1.61}$$

Amplitudes are complex numbers, all the spinor indices contracted, and can be written in terms of *spinor products*:

$$\langle pq\rangle = -\langle qp\rangle \quad , \quad [pq] = -[qp] \tag{6.1.62}$$

where:

$$\langle pq\rangle[qp] = \delta_{pq} = (p+q)^2 = 2p\cdot q. \tag{6.1.63}$$

Weyl spinors ($|p\rangle$, $|p]$) provide a complete representation of the Lorentz group $SO(1,3)$ for massless objects:

$$SO(1,3) \overset{p^2=0}{\longleftrightarrow} SU(2)\otimes SU(2). \tag{6.1.64}$$

Polarization vectors can be defined with respect to a reference direction n^μ satisfying:

$$\epsilon\cdot p = 0 \quad , \qquad \epsilon\cdot n = 0 \tag{6.1.65}$$

$$\epsilon_+\cdot\epsilon_- = -1 \quad , \qquad \epsilon_\pm\cdot\epsilon_\pm = 0 \tag{6.1.66}$$

$$\epsilon_+^\mu(p,n) = \frac{\langle n\sigma^\mu p]}{\sqrt{2}\langle np\rangle} \quad , \qquad \epsilon_-^\mu(p,n) = \frac{\langle p\sigma^\mu n]}{\sqrt{2}[np]}. \tag{6.1.67}$$

Exercise 7. Show the spin sum for the polarisation vectors is:

$$\sum_n \epsilon_n^\mu(p,n)\epsilon_n^\nu(p,n) = -g^{\mu\nu} + \frac{p^\mu n^\nu + p^\nu n^\mu}{p\cdot n} \tag{6.1.68}$$

which matches the propagator numerator in the light-like axial gauge:

$$\mathcal{L}_{gf} = \frac{1}{2\xi}(n\cdot A)^2. \tag{6.1.69}$$

We have also the *spinor product relations*:

$$\langle 12\rangle\langle 3| + \langle 23\rangle\langle 1| + \langle 31\rangle\langle 2| = 0 \tag{6.1.70}$$

$$\langle 12\rangle = \sqrt{|s_{12}|}e^{i\theta_{12}} \tag{6.1.71}$$

$$[12] = -\sqrt{|s_{12}|}e^{-i\theta_{12}} \tag{6.1.72}$$

$$\langle 1\sigma^\mu 2]\langle 3\sigma_\mu 4] = -2\langle 13\rangle[24] \tag{6.1.73}$$

where (6.1.70) is called *Schanten equation* and (6.1.73) is called *Fierz equation*. We are now ready to finish the partial amplitude computation for $q\bar{q}gg$. We have:

$$A_1 = -i\overline{u}\not\epsilon_4 \frac{\not p_{23}}{s_{23}} \not\epsilon_3 v_2 \tag{6.1.74}$$

$$A_3 = -i\overline{u}_1\gamma^\mu v_2 \frac{1}{s_{12}} \Big(\epsilon_3 \cdot \epsilon_4 (p_3 - p_4)_\mu + 2p_3 \cdot \epsilon_3 \epsilon_{4\mu} - 2p_3 \cdot \epsilon_4 \epsilon_{3\mu} \Big) \tag{6.1.75}$$

fermion currents (with incoming convention) must run from + to − and we can use parity symmetry to restrict 1^-2^+ as the only independent, non-zero, choice.

We can also talk about *helicity* $-+++$. With a bit of spinor algebra we find:

$$A_1 = \frac{2i\langle 1n_4\rangle[41]\langle 1n_3\rangle[32]}{s_{23}\langle n_3 3\rangle\langle n_4 4\rangle} \tag{6.1.76}$$

$$\begin{aligned} A_3 = \frac{-i\langle 1\sigma^\mu 2]}{\langle n_4 4\rangle\langle n_3 3\rangle s_{12}} \Big(&- 2\langle n_3 n_4\rangle[34](p_3 - p_4)_\mu + \langle n_4 \sigma_\mu 4]\langle n_3 \not 4 3]- \\ &- \langle n_3 \sigma_\mu 3]\langle n_4 \not 3 4]\Big) \end{aligned} \tag{6.1.77}$$

$$= \frac{-2i\langle 1n\rangle \Big(\langle n3\rangle[23] + \langle n4\rangle[24]\Big)[34]}{\delta_{12}\langle n_3 3\rangle\langle n_4 4\rangle} \tag{6.1.78}$$

where in the last step we fix $n_3 = n_4 = n$. So clearly for $n_3 = n_4 = 1$ all contributions vanish. We can see that *gauge invariance* implies Ward identity, which is biunivocal related with independency of reference vectors. So:

$$A(1^-_{\bar{q}} 2^+_q 3^+_g 4^+_g) = 0. \tag{6.1.79}$$

Exercise 8. Show that:

$$A(1^-_{\bar{q}} 2^+_q 3^+_g 4^+_g) = \frac{-2i\langle 31\rangle[24]^2}{\langle 12\rangle[12][23]} \tag{6.1.80}$$

$$= \frac{2i[24]^3[14]}{[12][23][34][41]} \tag{6.1.81}$$

$$= \frac{2i\langle 13\rangle^3\langle 23\rangle}{\langle 12\rangle\langle 23\rangle\langle 31\rangle\langle 41\rangle}. \tag{6.1.82}$$

6.1.4 Divergent graphs at 1-loop

The UV divergences are similar to those in QED (5.1.28) (founded via (3.1.14)), but with the difference that we need to add the ghosts contribution, so:

$$\omega(\Gamma_{QCD}) = 4 - \frac{3}{2}E_q - E_g - E_c. \tag{6.1.83}$$

You can see the table 6.1.

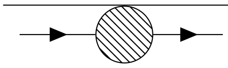
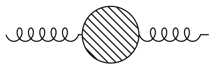
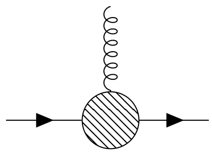
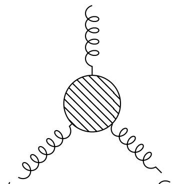
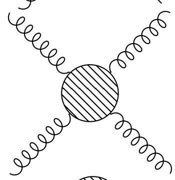
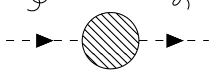
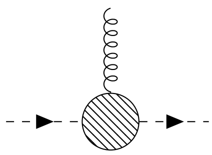
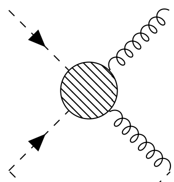
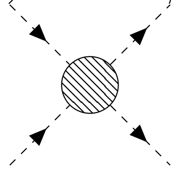
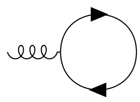
Γ	n_q	n_g	n_c	$\omega(\Gamma_{QED})$	name
	2	0	0	1	quark self-energy
	0	2	0	2	gluon self-energy
	2	1	0	0	$q\bar{q}g$-vertex
	0	3	0	1	$3g$-vertex
	0	4	0	0	$4g$-vertex
	0	0	2	2	ghost self-energy
	0	1	2	1	$c\bar{c}g$-vertex
	0	2	2	0	-
	0	0	4	0	-

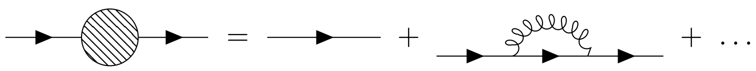
Table 6.1: Divergent diagrams of QCD.

In the table we have ommitted the tadpole graph:

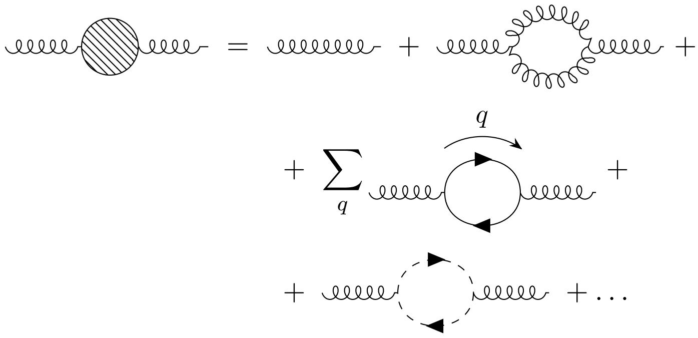


which is trivially zero (by color algebra this time, and not for gammas' properties and the scalelessness integral). The last two graphs do not match gauge invariant operators and must therefore give *finite results* when all diagrams are summed together.

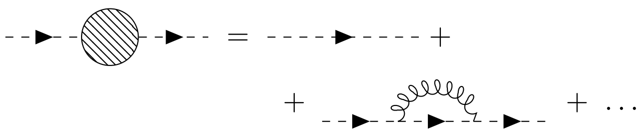
The major difference with respect to QED is the number of diagrams that contribute for every graph, due to the higher number of fields present in the theory. Up to one-loop we have:



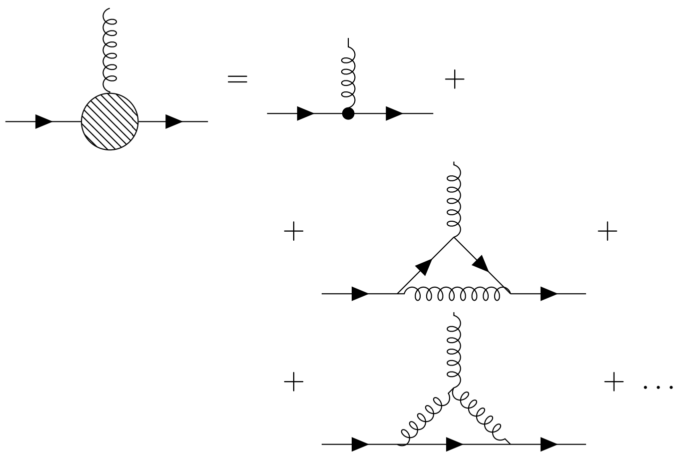
(6.1.84)



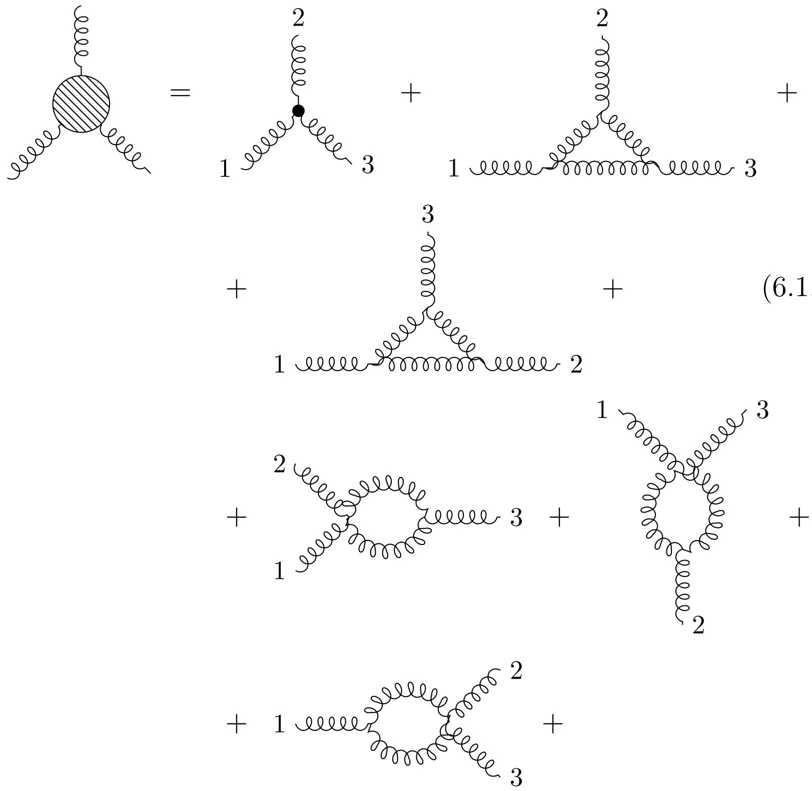
(6.1.85)



(6.1.86)



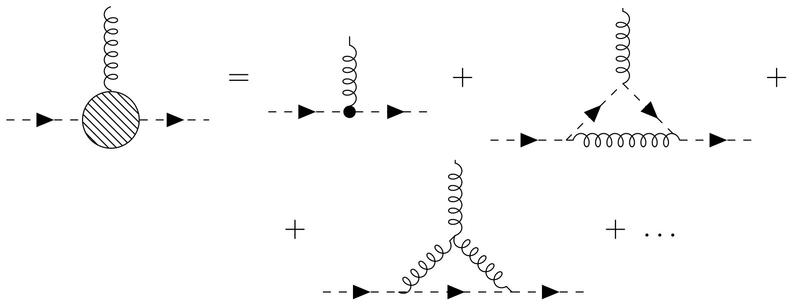
(6.1.87)



(6.1.88)

+ ghost loops + quark loops + (higher order)

(6.1.89)



(6.1.90)

In the same way 4−point diagram expansions can be obtained relatively easily if required, clearly there are quite a few diagrams to consider.

6.1.5 Renormalization counter-terms

The necessary field and parameter redefinitions are as follows:

$$\psi_{q0} = Z_{\psi_q}^{1/2} \psi_{qR} \quad , \qquad A_0^\mu = Z_A^{1/2} A_R^\mu \tag{6.1.91}$$

$$c_0 = Z_c^{1/2} c_R \quad , \qquad m_{q0} = Z_{m_q} m_{qR} \tag{6.1.92}$$

$$g_{s_0} = g_{s_R} \mu_R^\epsilon Z_{g_s} . \tag{6.1.93}$$

For the rest of this chapter quantities without R subscripts will be considered to be renormalized, e.g. $\psi_q = \psi_{qR}$. The QCD lagrangian in terms of renormalized quantities has the following counter-terms:

$$
\begin{aligned}
\mathcal{L}_{QCD} = \mathcal{L}_{QCD,R} + & \\
& + \sum_q \Big[i\overline{\psi}_q \partial\!\!\!/ \psi_q (Z_{\psi_q} - 1) + m_q \overline{\psi}_q \psi_q (Z_{m_q} Z_{\psi_q} - 1) - \\
& - g_s \mu_R^{\epsilon} \overline{\psi}_q A\!\!\!/^a t^a \psi_q (Z_1 - 1) - \\
& - \overline{c}^a \Box c^a (Z_c - 1) - g_s \mu_R^{\epsilon} f^{abc} \partial_\mu \overline{c}^a A^{\mu b} c^c (Z_1^c - 1) + \\
& + \frac{1}{2} A_\mu^a (g^{\mu\nu} \Box - \partial^\mu \partial^\nu) A_\nu^a (Z_A - 1) + \\
& + g_s \mu_R^{\epsilon} f^{abc} A^{\mu b} A^{\nu c} \partial_\mu A_\nu^a (Z_1^{3g} - 1) - \\
& - \frac{1}{4} g_s^2 \mu_R^{2\epsilon} f^{abx} f^{xcd} A_\mu^a A_\nu^b A^{\mu c} A^{\nu d} (Z_1^{4g} - 1) \Big]
\end{aligned} \tag{6.1.94}
$$

where we define:

$$
Z_1 = Z_{g_s} Z_A^{1/2} Z_{\psi_q} \quad , \qquad Z_1^c = Z_{g_s} Z_A^{1/2} Z_c \tag{6.1.95}
$$

$$
Z_1^{3g} = Z_{g_s} Z_A^{3/2} \quad , \qquad Z_1^{4g} = Z_{g_s}^2 Z_A^2. \tag{6.1.96}
$$

Unlike QED the gauge invariance constraint on Z_x do not imply $Z_1 = Z_{\psi_q}$. This time there are a set of relations for correlation functions that follow form gauge invariance called *Slavrov-Taylor identities*; in turn we may derive additional constraints on Z_x, but we cannot avoid the vertex computation.

Indeed, we can notice that the coupling constant is present in 4 different diagrams, so since it cannot be renormalized in 4 different way, because the renormalization has to give the same result for different process, we obtain the relations:

$$
Z_g \sqrt{Z_A} = \frac{Z_1}{Z_\psi} \quad , \qquad Z_g \sqrt{Z_A} = \frac{Z_1'}{Z_A} \tag{6.1.97}
$$

$$
Z_g \sqrt{Z_A} = \frac{Z_1''}{Z_c} \quad , \qquad Z_g \sqrt{Z_A} = \sqrt{\frac{Z_4}{Z_A}}. \tag{6.1.98}
$$

So we have the scheme in figure 6.1, that compares QED and QCD.

6.1.6 Strong CP problem

We finish this section with a small remark on the form of the QCD lagrangian. If we were to write down all $SU(N_c)$ guage invariant operators in 4 dimensions we would include:

$$
F^{\mu\nu a} \tilde{F}_{\mu\nu}^a \tag{6.1.99}
$$

where:

$$
\tilde{F}_{\mu\nu}^a = \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma a}. \tag{6.1.100}
$$

$$\begin{tikzcd}
& U(1)\text{ gauge invariance} \arrow[dl] \arrow[dr] & \\
Z_1 = Z_2 & & \begin{array}{c}\text{Ward-Takahashi}\\ \text{Identity}\end{array} \arrow[ll]
\end{tikzcd}$$

(a) Scheme for QED.

$$\begin{tikzcd}
& SU(N_c)\text{ gauge invariance} \arrow[dl] \arrow[dr] & \\
\begin{array}{c}\dfrac{Z_1}{Z_{\psi_q}} = \dfrac{Z_1^c}{Z_c}\\ + \text{ other relations}\end{array} & & \begin{array}{c}\text{Slaunov-Taylor}\\ \text{Identity}\end{array} \arrow[ll]
\end{tikzcd}$$

(b) Scheme for QCD.

Figure 6.1

This term represents a subtle problem in QCD, in which its presence implies CP violating strong interaction which are not observed experimental.

Exercise 9. Show that:

$$F^{\mu\nu a}\tilde{F}^a_{\mu\nu} = \partial_\mu k^\mu \tag{6.1.101}$$

$$k^\mu = \epsilon^{\mu\nu\rho\sigma} A^a_\nu \left(F^a_{\rho\sigma} - \frac{g_s}{3} f^{abc} A^b_\rho A^c_\sigma \right). \tag{6.1.102}$$

Fortunately, in the exercise we see the operator $F^{\mu\nu a}\tilde{F}^a_{\mu\nu}$, which is in the lagrangian, is a *total derivative*, so plays no role in perturbation theory.

We can notice that QED (abelian gauge theory) could also have a term $F^{\mu\nu}\tilde{F}_{\mu\nu}$, but this total derivative has no non-perturbative conseguences.

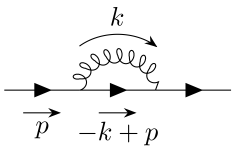
't Hooft showed that the QCD (non-abelian gauge theory) vacuum has *istanton* contributions, so $F\tilde{F}$ does have non-perturbative conseguences. Indeed, a proposed solution to $F\tilde{F}$ contribution is to introduce a new dynamical field, the *axion*, which can naturally tune the coefficient of $F\tilde{F}$ to be very small (undetectable). This is the *Peccei-Quinn mechanism.* So far the axions have not been confirmed in experiment either.

6.2 One-loop graph computations

In this section we'll see the calculation of one-loop graph, in order to determine the counterterms to renormalize the theory.

6.2.1 Quark self-energy

There is only one diagram and the only difference with respect to QED is the color factor:



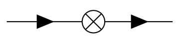
$$= t^a_{ik} t^a_{kj} g_s^2 \mu_R^{2\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\frac{\gamma^\mu(-\not{k} + \not{p} + m_q)\gamma_\mu}{D(k,0)D(k-p,m_q)} \right) \tag{6.2.1}$$

$$= C_F \delta_{ij} \left(\frac{\alpha_s}{4\pi}\right) i C_\Gamma \left(\frac{1}{\epsilon}(\not{p} - 4m_q) + \mathcal{O}(\epsilon^0) \right) \tag{6.2.2}$$

where we use:

$$\alpha_s = 4\pi g_s^2 \quad , \quad C_\Gamma = (4\pi)^\epsilon \Gamma(1+\epsilon). \tag{6.2.3}$$

We can see the contribution:



$$= i\delta_{ij} \Big(\delta_{\psi_q} \not{p} - \delta_3 m_q \Big) \tag{6.2.4}$$

$$= i\delta_{ij} \Big(\delta_{\psi_q}(\not{p} - m) - \delta_{m_q} m_q \Big). \tag{6.2.5}$$

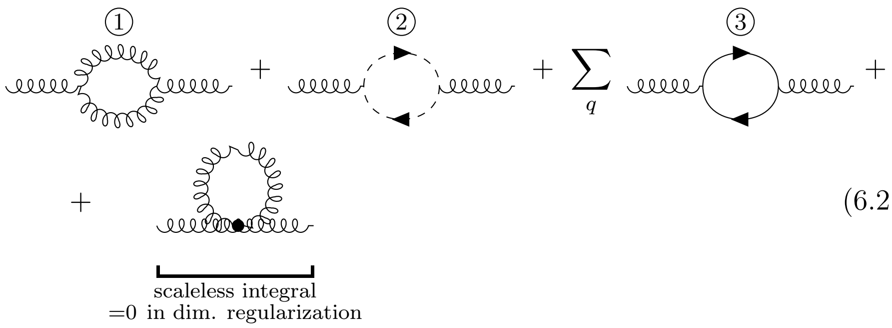
Since for the strong interaction there is no perturbative low energy limit, we cannot perform an on-shell renormalization and so we use $\overline{MS}$:

$$\delta^{(1)}_{\psi_q} = \left(\frac{\alpha_s}{4\pi}\right) C_\Gamma \left(-\frac{1}{\epsilon}\right) C_F \tag{6.2.6}$$

$$\delta^{(1)}_{m_q} = \left(\frac{\alpha_s}{4\pi}\right) C_\Gamma \left(-\frac{3}{\epsilon}\right) C_F. \tag{6.2.7}$$

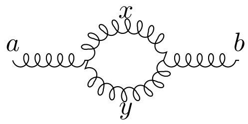
6.2.2 Gluon self-energy

The QCD computation differs from the QED analogue in that there are more diagrams for each process, and the ghost diagrams are required to produce the correct transverse tensor structure dictated by gauge invariance. We are saying:



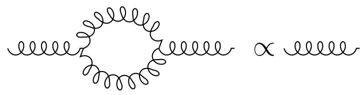
(6.2.8)

Let's start with the color part of the gluon diagram:



$$= f^{axy} f^{byx}. \tag{6.2.9}$$

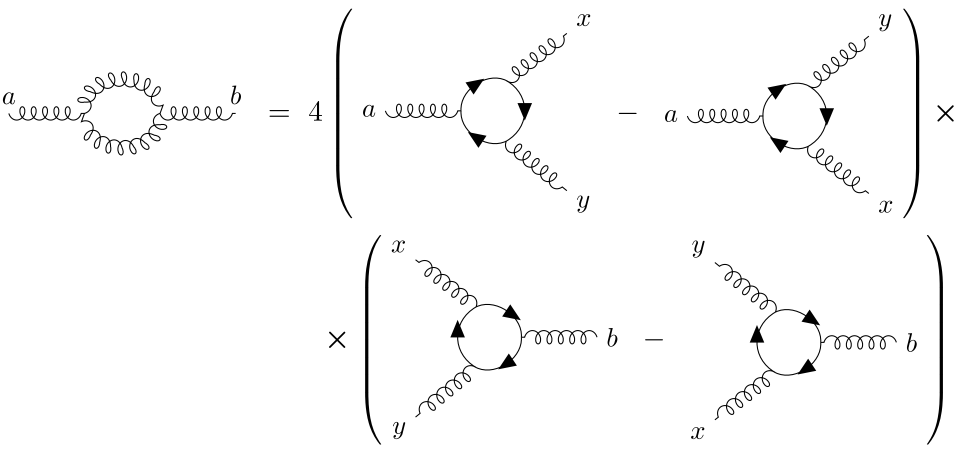
We may already anticipate that this could be proportional to the tree-level color factor δ^{ab}:



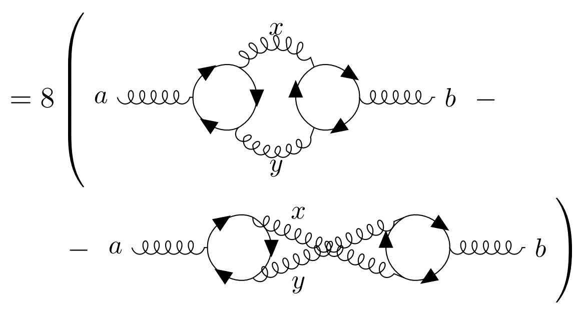
Indeed, we can write:

$$f^{abc} = -2i\,\mathrm{Tr}\Big\{\Big[t^a, t^b\Big]t^c\Big\} \tag{6.2.10}$$

and applying the Fierz identity we can derive the constant of proportionality:

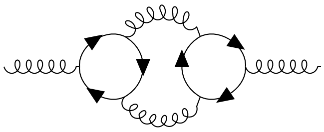
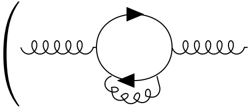


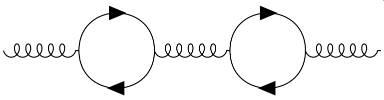
(6.2.11)

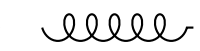


(6.2.12)

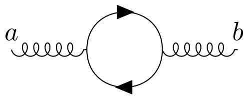
the first term is:

 $= \frac{1}{2}\Bigg($  $-$

$- \frac{1}{N_c}$  $\Bigg)$ (6.2.13)

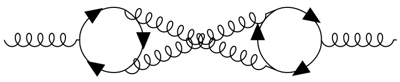
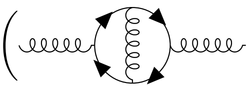
$= \frac{1}{4}\left(C_F - \frac{1}{2N_c}\right)$  (6.2.14)

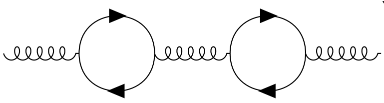
where:

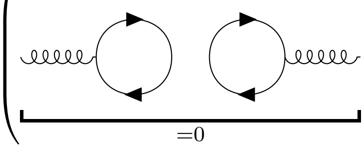


$= \mathrm{Tr}\Big\{t^a t^b\Big\} = \frac{1}{2}\delta^{ab}.$ (6.2.15)

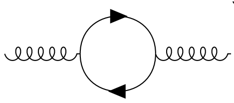
While the second term is:

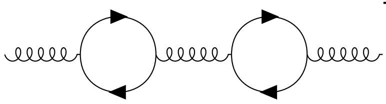
 $= \frac{1}{2}\Bigg($  $-$

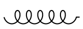
$- \frac{1}{N_c}$  $\Bigg)$ (6.2.16)

$= \frac{1}{2}\Bigg[\frac{1}{2}\Bigg($ 

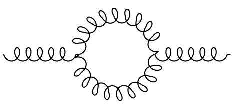
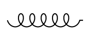
$-$

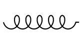
$- \frac{1}{N_c}$  $\Bigg) -$

$- \frac{1}{N_c}$  $\Bigg]$ (6.2.17)

$= -\frac{1}{4N_c}$  (6.2.18)

So we determine:

 $= 8\left(\frac{1}{4}\frac{N_c^2-1}{2N_c} - \frac{1}{8N_c} - \frac{1}{4N_c}\right)$  (6.2.19)

$= -N_c$  (6.2.20) where we identify the adjoint Casimir:

$$C_A = N_c. \tag{6.2.21}$$

Exercise 10. We can prove this relation using an alternative method which introduces the symmetric structure constants d^{abc}. Those are the steps:

- Using the definition:

$$\left\{t^a, t^b\right\} = d^{abc} t^c + \frac{1}{N_c} \delta^{ab} \tag{6.2.22}$$

  show:

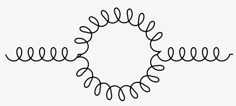
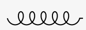
$$d^{abc} = 2\,\mathrm{Tr}\left\{\left\{t^a, t^b\right\} t^c\right\}. \tag{6.2.23}$$

- Multiply the relation:

$$\frac{1}{4}\left(d^{axy} + i f^{axy}\right) = \mathrm{Tr}\{t^a t^x t^y\} \tag{6.2.24}$$

  with f^{byx} to find a relation for $f^{axy} f^{byx}$ in terms of traces.
  NB: $d^{axy} f^{byx} = 0$.

- Compute traces to arrive at:

 $= -C_A$  (6.2.25)

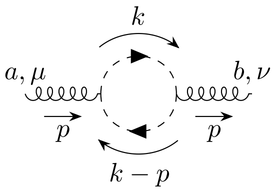
The color factor for the other diagrams are now clear as well (the subscript $1, 2, 3$ are related to the number in (6.2.8)):

$$d_1 = -C_A K_1 \delta_{ab} \tag{6.2.26}$$
$$d_2 = -C_A K_2 \delta_{ab} \tag{6.2.27}$$
$$d_3 = \frac{1}{2} K_3 \delta_{ab}. \tag{6.2.28}$$

The kinematic factor K_3 is identical to the photon self-energy, so we may immediately arrive at:

$$K_3 = i\left(\frac{\alpha_s}{4\pi}\right) C_\Gamma \Big(- g^{\mu\nu} p^2 + p^\mu p^\nu \Big) \left(-\frac{4}{3\epsilon} + \mathcal{O}(\epsilon^0) \right). \tag{6.2.29}$$

The ghost loop is perhaps the next simplest:



$$= -C_A\delta_{ab}\Big(i^2 g_s^2\mu_R^{2\epsilon}\Big)\int_k \frac{k^\mu (k-p)^\nu}{D(k,0)D(k-p,0)} \tag{6.2.30}$$

$$= -C_A\delta_{ab}K_2 \tag{6.2.31}$$

where we defined the coefficient K_2 as the prefactor and the integral in k.

Exercise 11. Show:

$$K_2 = -g_s^2\mu_R^{2\epsilon}\Big(p^2 g^{\mu\nu} + (d-2)p^\mu p^\nu\Big)\frac{1}{4(d-1)}I_{2,00}^{[4-2\epsilon]}[1] \tag{6.2.32}$$

$I_{2,00}^{[4-2\epsilon]}[1]$ massless bubble integral computed in Chapter §3.

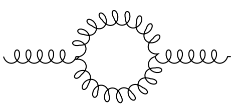
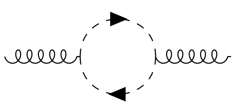
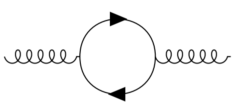
Exercise 12. Show the gluon diagram results in:

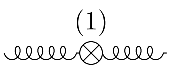
$$K_3 = -\frac{1}{2}g_s^2\mu_R^{2\epsilon}\Big[-(6d-5)p^2 g^{\mu\nu} + (7d-6)p^\mu p^\nu\Big]\times$$
$$\times\frac{1}{2(d-1)}I_{2,00}^{[4-2\epsilon]}[1]. \tag{6.2.33}$$

Exercise 13. Show that the sum of all diagrams gives:

$$d_1 + d_2 + \sum_q d_3 = \left(\frac{\alpha_s}{4\pi}\right)C_\Gamma\Big(-p^2 g^{\mu\nu} + p^\mu p^\nu\Big)\times$$
$$\times\frac{1}{\epsilon}\left(\frac{5}{3}C_A - \frac{2}{3}n_f\right) + \mathcal{O}(\epsilon^0). \tag{6.2.34}$$

Combining with te counterterm we determine $\delta_A^{(1)}$:

 $+$  $+ \sum_q$  $+$

$+$ 

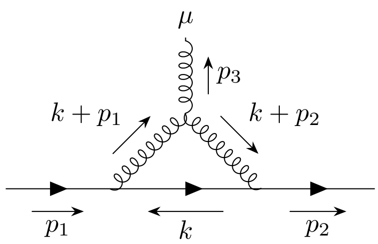
$= \mathcal{O}(\epsilon^0)$ (6.2.35)

and using $\overline{MS}$, so we get:

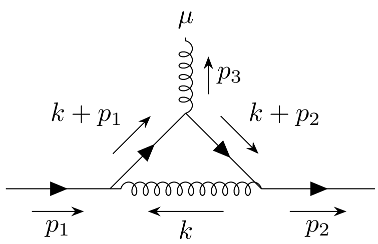
$$\delta_A^{(1)} = \left(\frac{\alpha_s}{4\pi}\right)C_\Gamma\left(\frac{5C_A}{3} - \frac{2n_f}{3}\right)\frac{1}{\epsilon}. \tag{6.2.36}$$

6.2.3 The quark-gluon vertex

The two contributing one-loop diagrams are:

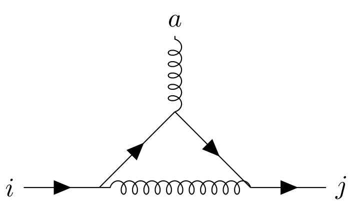


$+$

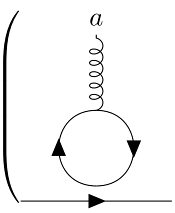


$$d_1 = C_1 K_1 \qquad d_2 = C_2 K_2. \tag{6.2.37}$$

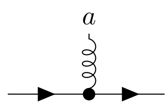
The color factors are simple to determine:



$= \frac{1}{2} \Bigg($

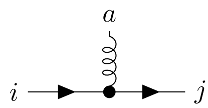


$- \frac{1}{N_c}$



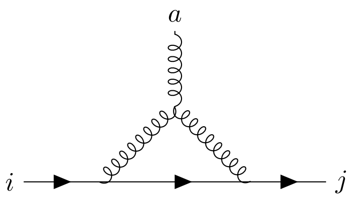
$\Bigg)$ (6.2.38)

$= -\frac{1}{2N_c}$

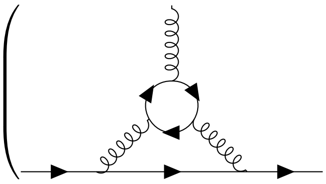


(6.2.39)

$$= C_2 \tag{6.2.40}$$



$= \Bigg($



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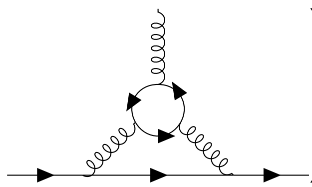
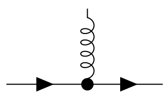


$\Bigg) (-2i)$ (6.2.41)

$= -i\frac{N_c}{2}$



(6.2.42)

$$= C_1 \tag{6.2.43}$$

where the step (6.2.42) is left as exercise.

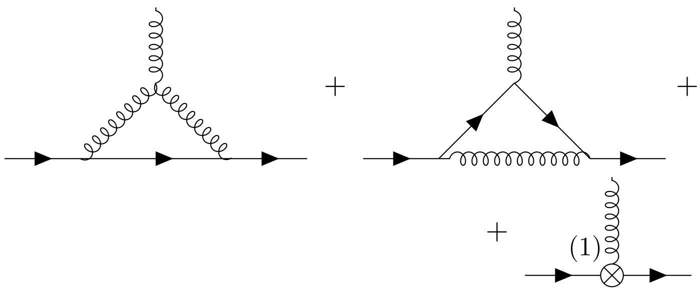
Exercise 14. Show the kinematic contributions are:

$$K_1 = \left(\frac{\alpha_s}{4\pi}\right) C_\Gamma i g_s \overline{u}_2 \gamma^\mu v_1 \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0) \tag{6.2.44}$$

$$K_2 = \left(\frac{\alpha_s}{4\pi}\right) C_\Gamma i g_s \overline{u}_2 \gamma^\mu v_1 \left(\frac{3i}{\epsilon}\right) + \mathcal{O}(\epsilon^0) \tag{6.2.45}$$

where the IR divergences in the scalar integrals $I^{[4-2\epsilon]}_{3,0mm}$ and $I^{[4-2\epsilon]}_{3,000}$ have been discorded.

Combining with the contribution from the vertex counter-term we find ($\overline{MS}$):



$$= \mathcal{O}(\epsilon^0) \tag{6.2.46}$$

so:

$$\delta_1^{(1)} = \left(\frac{\alpha_s}{4\pi}\right) C_\Gamma \left(-\frac{3N_c}{2} + \frac{1}{2N_c}\right)\left(\frac{1}{\epsilon}\right) \tag{6.2.47}$$

$$= \left(\frac{\alpha_s}{4\pi}\right) C_\Gamma \left(C_A + C_F\right)\left(-\frac{1}{\epsilon}\right). \tag{6.2.48}$$

6.3 Renormalization of the strong coupling

We saw in the section §5.4 the vaccum polarization phenomenon (an screening effect) for the QED. Now, we can see the analogue for the QCD, but we will see that these theories are profoundly different. All these results will be further inverstigated in the chapter §7 where we'll talk about the running coupling, calculating the beta functions of the theories.

From the relation:

$$Z_1 = Z_{g_s} Z_A^{1/2} Z_{\psi_q} \tag{6.3.1}$$

we can obtain:

$$\delta_1^{(1)} - \delta_{\psi_q}^{(1)} - \frac{1}{2}\delta_A^{(1)} = \delta_{g_s}^{(1)} \tag{6.3.2}$$

hence:

$$\delta_{g_s}^{(1)} = \left(\frac{\alpha_s}{4\pi}\right) C_\Gamma \left[-\frac{1}{\epsilon}(C_A + C_F) - \left(-\frac{1}{\epsilon}\right) - \frac{1}{2}(5C_A - 2n_f)\frac{1}{3\epsilon}\right] \tag{6.3.3}$$

$$= -\left(\frac{\alpha_s}{4\pi}\right) C_\Gamma \left(\frac{11C_A - 2n_f}{6\epsilon}\right) \tag{6.3.4}$$

to compare this directly with QED ($U(1)$) we can identify the quark loop with a factor of T_R, where $T_R = 1/2$ for QCD and $T_R = 1$ for QED, so:

$$\delta_{g_s}^{(1)} = -\left(\frac{\alpha_s}{4\pi}\right) C_\Gamma \left(\frac{11C_A - 4T_R n_f}{6\epsilon}\right) \tag{6.3.5}$$

$$\delta_e^{(1)} = \delta g_s^{(1)}\Big|_{\substack{C_A \to 0 \\ T_R n_f \to 1 \\ \alpha_s \to \alpha}} \tag{6.3.6}$$

$$= \left(\frac{\alpha_s}{4\pi}\right) C_\Gamma \frac{2}{3\epsilon}. \tag{6.3.7}$$

The charge screening effect seen in QED implied that:

$$\alpha_{eff}(Q^2) > \alpha_{eff}(0) \tag{6.3.8}$$

for a scale $Q^2 > 0$. The analogous effect in QCD will depend on the sign of δ_{g_s}. For $N_c = 3$ there is a sign change (with respect to QED) if:

$$33 - 2n_f > 0 \tag{6.3.9}$$

hence:

$$n_f \leq 16 \tag{6.3.10}$$

remembering that n_f must be integer. The Standard Model has $n_f = 6$ and so:

$$\alpha_{s,eff}(Q^2) < \alpha_{s,eff}(Q'^2) \tag{6.3.11}$$

for:

$$Q^2 > Q'^2. \tag{6.3.12}$$

So, while the QED $\alpha_{eff}(Q^2)$ increases with the scale, the QCD $\alpha_{eff}(Q^2)$ decreases. As already said we will see again this behavior with the beta functions of the different theory.

Chapter 7

The renormalization group

In this chapter, after a brief historical rewiew, we show how the renormalization procedure causes model parameters to evolve with energy scale. We see how the β function describes how the coupling runs with energy. Also, we see how the Callan-Symanzik equation follows from the fact the renormalization scale dependence is a conseguence of absorbing UV divergences.

The main reference for this chapter is Peskin Let no one ignorant of geometry enter my door.Peskin.

7.1 History

After the renormalization of QED, the systematics of the procedure and conseguences on physical observables were still not fully understood.

Between 1950-1970 there were several studies of how residual scale dependence on *renormalized quantities* in perturbation theory implied *evolution equations* for model parameters (coupling, masses etc.). The first works were by *Stuekelberg, Peterman* (1953) and *Gell-Mann, Low* (1954) studying electric charge in QED. Same ideas came from Condensed matter systems (*Kadanoff*, 1966). The ideas were formalized in the works of *Callan-Symanzik* (1970) and *Wilson* (1971-1975).

Indeed, the troublesome short-distance singularities, which forced upon us the great machinery of renormalization, turn out to be the key to scaling properties of operators. A set of differential equations may be written expressing the nontrivial effect of an infinitesimal change in the scale. Upon integration they reveal that the short-distance behavior acquires, in favorable cases, a universal character, with fields and composite operators being assigned anomalous dimensions.

These ideas find successful applications to deep inelastic lepton scattering, electron-positron annihilation, and other high-energy processes. The concepts of short-distance expansion, asymptotic freedom, enrich the theorist's arsenal, and motivate hopes of developing a fundamental theory of strong interactions.

The parallel development in statistical mechanics (my notes on *Statistical Mechanics - The renormalization group* are on my personal page) has been tremendous, and has achieved notable successes in the comparison with experimental facts.

There were several Nobel prize for these works: Feynman, Schwinger, Tomonaga (1965), Gell-Mann (1969), Glashow, Salam, Weinberg (1979), Wilson (1982), 't Hooft, Veltman (1999), Gross, Politter, Wilczek (2004).

7.1.1 Scale dependence of electric charge

Relations between large-momentum or short-distance behavior and renormalizability properties have been discussed in the early 1950s by Stueckelberg and Peterman and by Gell-Mann and Low. We talked about these properties in sections §5.4 and §6.3.

Let's see the work by Gell-Mann and Low. The reference for this section is Itzykson-Zuber Let no one ignorant of geometry enter my door.Zuber at page 633-635.

We can use our result for $\Pi^{\mu\nu}$ (5.2.71) at one-loop to study the effective charge at different scales. Recall also (5.2.74):

$$\tilde{G}_{\mu\nu}(q^2, \dots) = \frac{-ig^{\mu\nu}}{q^2} \frac{1}{1 + \Pi(q^2, \dots)} + G_L^{\mu\nu}(q^2, \dots).$$

The renormalized vacuum polarization is a function of q^2, α and the mass m:

$$\Pi(\alpha, q^2, m^2) \tag{7.1.1}$$

Π was renormalized on-shell through the condition:

$$\Pi(\alpha, 0, m^2) = 0. \tag{7.1.2}$$

We can define an *effective charge* at the scale q^2, called $d(\alpha, q^2, m^2)$, by considering the combination $\alpha\tilde{G}_{\mu\nu}$ associated to photon propagation:

$$\alpha\tilde{G}^{\mu\nu} = \frac{-ig^{\mu\nu}}{q^2} d(\alpha, q^2, m^2) + \alpha\tilde{G}_L^{\mu\nu} \tag{7.1.3}$$

which implies:

$$d(\alpha, q^2, m^2) = \frac{\alpha}{1 + \Pi(\alpha, q^2, m^2)} \tag{7.1.4}$$

and at $q^2 = 0$:

$$d(\alpha, q^2 = 0, m^2) = \alpha. \tag{7.1.5}$$

We could ask ourselves what happens if we fix the charge at a different point, say at $q^2 = \lambda^2$, with $\lambda^2 < 0$ (so space-like, in euclidean region away from singularities). The perturbation series will be defined in terms of a parameter α_λ equal to:

$$\alpha_\lambda = d(\alpha, \lambda^2, m^2) = \frac{\alpha}{1+\Pi(\alpha, \lambda^2, m^2)} \tag{7.1.6}$$

we can now re-write α in terms of α_λ such that:

$$d(\alpha, q^2, m^2) = D(\alpha_\lambda, q^2, m^2; \lambda^2) \tag{7.1.7}$$

where the new expression for the effective charge satisfies:

$$\alpha_\lambda = D(\alpha_\lambda, \lambda^2, m^2; \lambda^2). \tag{7.1.8}$$

Equation (7.1.7) expresses the apparently empty fact that the physical content is not modified by a mere change in the conventions. A different subtraction point leads to a modification of $\aleph_\lambda$ preserving the function D.

Since d, the effective charge, is dimensionless we can rewrite (7.1.7), in region $q^2 < 0$, $m^2 > 0$ and $\lambda^2 < 0$, as:

$$d(\alpha, -q^2/m^2) = D(\alpha_\lambda, q^2/\lambda^2; -\lambda^2/m^2) \tag{7.1.9}$$

so, only in terms of ratio of dimensional parameters, and in which (7.1.8) became:

$$\alpha_\lambda = d(\alpha, -\lambda^2/m^2) = D(\alpha_\lambda, 1, -\lambda^2/m^2). \tag{7.1.10}$$

We now study (7.1.9) in the asymptotic limit $-q^2/m^2 \to \infty$ (which implies $\alpha \log\left(-\frac{q^2}{m^2}\right) \to 0$), which gives us:

$$\Pi(\alpha, -q^2/m^2) \overset{-q^2/m^2\to\infty}{\longrightarrow} \Pi^{asy}(\alpha, -q^2/m^2). \tag{7.1.11}$$

We computed this directly in §5 (see (5.4.7)) in dimensional regularization:

$$\Pi^{asy}_{\overline{MS}}(\alpha, -q^2/m^2) = \left(\frac{\alpha}{4\pi}\right) C_\Gamma \frac{4}{3}\left(-\frac{5}{3} + 3\log\left(-\frac{q^2}{m^2}\right)\right) + \mathcal{O}\left(\alpha^2\right)$$

where we see one argument only in the logarithm, so we can show that:

$$D(\alpha_\lambda, q^2/\lambda^2; -\lambda^2/m^2) \overset{-q^2/m^2\to\infty}{\longrightarrow} D^{asy}\left(\alpha_\lambda, \left(-q^2/m^2\right)\left(-m^2/\lambda^2\right)\right) \tag{7.1.12}$$

$$d\left(\alpha, -q^2/m^2\right) \overset{-q^2/m^2\to\infty}{\longrightarrow} d^{asy}\left(\alpha, -q^2/m^2\right) \tag{7.1.13}$$

so that the asymtotic limit of (7.1.9) is:

$$d^{asy}(\alpha, x) = D^{asy}(\alpha_y, x/y) \tag{7.1.14}$$

where:

$$x = -\frac{q^2}{m^2} \quad , \quad y = -\frac{\lambda^2}{m^2} \tag{7.1.15}$$

and:

$$\alpha_y = d^{asy}(\alpha, y) = D^{asy}(\alpha_y, 1). \tag{7.1.16}$$

The *Gell-Mann, Low function* is defined as:

$$\psi(z) = \frac{\partial}{\partial x} D^{asy}(z, x)\bigg|_{x=1} \tag{7.1.17}$$

and we can use (7.1.9) to prove:

$$\psi(\alpha_x) = x\frac{\partial}{\partial x} d^{asy}(\alpha, x). \tag{7.1.18}$$

Proof. Starting with:

$$\begin{aligned}
\frac{\partial}{\partial x} d^{asy}(\alpha, x) &= \frac{\partial}{\partial x} D^{asy}(\alpha_y, x/y) &(7.1.19)\\
&= \frac{\partial(x/y)}{\partial x}\frac{\partial}{\partial(x/y)} D^{asy}(\alpha_y, x/y) &(7.1.20)\\
&= \frac{1}{y}\frac{\partial}{\partial z} D^{asy}(\alpha_y, z) &(7.1.21)
\end{aligned}$$

writing $z = x/y$. The LHS (Left-Hand-Side) of this equation is independent of y, so we may set $y = x$ to find:

$$\begin{aligned}
\frac{\partial}{\partial x} d^{asy}(\alpha, x) &= \frac{1}{x}\left(\frac{\partial}{\partial x} D^{asy}(\alpha_x, x)\right)_{x=1} &(7.1.22)\\
&= \frac{1}{x}\psi(\alpha_x) &(7.1.23)
\end{aligned}$$

which is our statement. ■

The solution of the Gell-Mann and Low equation (7.1.18) shows that it is possible to obtain the dominant behavior of the vacuum polarization to order n, knowing d^{asy} and therefore ψ to lower orders; i.e. we can say that the function $\psi(\alpha)$ determines the evolution of the effective charge (in the asymptotic region). In the end what we need to keep in mind is:

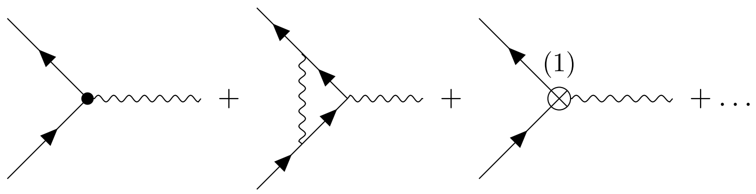
- $\tilde{G}^{\mu\nu}$ does not change, and it is independent of the renormalization scheme.
- The effective charge does change when we change the scale, at which we fixed the renormalization scheme. The scale dependence is given by the Gell-Mann Low function $\psi(x)$.

Callan-Symanzik clarified and generalized this picture.

7.2 Scale dependence of perturbative predictions

Consider the perturbative corrections to the vertex function which can be related to a physical observable O (in QED):

$O(q^2) \sim$



$$= \lambda O^{(1)}(q^2) + \lambda^3 O^{(2)}(q^2;\mu_R^2). \tag{7.2.1}$$

Let's compare this to an experiment at q_0^2, where we measure $O(q^2) = \sigma$ and so, at leading order (LO):

$$\sigma = \lambda\sigma^{(1)}(q_0^2) + \mathcal{O}(\lambda^3) \tag{7.2.2}$$

$$\implies \quad \lambda_{LO} = \frac{\sigma}{\sigma^{(1)}(q_0^2)}. \tag{7.2.3}$$

If we compare theory and experiment at next-to-leading-order (NLO):

$$\sigma = \lambda\sigma^{(1)}(q_0^2) + \lambda^3\sigma^{(2)}(q_0^2;\mu_R^2) + \mathcal{O}(\lambda^5) \tag{7.2.4}$$

the measurement has not changed, but the value of the coupling has:

$$\lambda_{LO}\sigma^{(1)}(q_0^2) = \lambda_{NLO}\sigma^{(1)}(q_0^2) + \lambda_{NLO}^3\sigma^{(2)}(q_0^2;\mu_R^2). \tag{7.2.5}$$

For a single scale process, such as this one, the NLO correction will take the form:

$$\sigma^{(2)}(q_0^2;\mu_R^2) = \sigma^{(1)}(q_0^2)\left(a\log\left(\frac{\mu_R^2}{q_0^2}\right) + b\right) \tag{7.2.6}$$

and so:

$$\lambda_{LO} = \lambda_{NLO} + \lambda_{NLO}^3\left(a\log\left(\frac{\mu_R^2}{q_0^2}\right) + b\right). \tag{7.2.7}$$

7.3 The Callan-Symanzik equation

At the end of Chapter §3, in the section §3.6.5, we considered the scale dependence of the renormalized correlation functions. You can see the chapter §12.2 from Peskin Let no one ignorant of geometry enter my door.Peskin for detail. We've already saw that comparing bare and renormalized functions implied:

$$\begin{aligned}\tilde{G}_0(p_1,\ldots,p_n;\lambda_0) &= FT\left(\langle 0|\, T\{\phi_0(x_1)\ldots\phi_0(x_n)\}\,|0\rangle\right) &(7.3.1)\\ &= Z_\phi^{n/2}(\mu_R,\lambda_R(\mu_R))\tilde{G}_R(p_1,\ldots,p_n;\lambda_R(\mu_R),\mu_R). &(7.3.2)\end{aligned}$$

From this expression we derive (imposing that $\tilde{G}_0$ does not depends on the scale μ_R):

$$
\begin{aligned}
0 &= \mu_R \frac{\mathrm{d}}{\mathrm{d}\mu_R} \tilde{G}_0 && (7.3.3)\\
&= \mu_R \frac{\mathrm{d}}{\mathrm{d}\mu_R} \left(Z_\phi^{n/2} \tilde{G}_R \right) && (7.3.4)\\
&= \mu_R \left(\frac{\mathrm{d}}{\mathrm{d}\mu_R} Z_\phi^{n/2} \right) \tilde{G}_R + \mu_R Z_\phi^{n/2} \left(\frac{\mathrm{d}}{\mathrm{d}\mu_R} \tilde{G}_R \right) && (7.3.5)\\
&= n (Z_\phi^{1/2})^{n-1} \mu_R \left(\frac{\mathrm{d}}{\mathrm{d}\mu_R} Z_\phi^{1/2} \right) \tilde{G}_R + Z_\phi^{n/2} \mu_R \frac{\mathrm{d}}{\mathrm{d}\mu_R} \tilde{G}_R && (7.3.6)\\
&= Z_\phi^{n/2} \left[\mu_R \frac{\mathrm{d}}{\mathrm{d}\mu_R} \tilde{G}_R + n \tilde{G}_R \mu_R \frac{\mathrm{d}}{\mathrm{d}\mu_R} \log \left(Z_\phi^{1/2} \right) \right]. && (7.3.7)
\end{aligned}
$$

For a *minimal scheme* (so MS and $\overline{MS}$ etc.) we may state:

$$Z_\phi \equiv Z_\phi(\lambda_R(\mu_R)) \tag{7.3.8}$$

and so the relation becomes:

$$
\begin{aligned}
\mu_R \frac{\partial}{\partial \mu_R} \tilde{G}_R(\mu_R, \lambda(\mu_R)) + \mu_R \frac{\mathrm{d}\lambda(\mu_R)}{\mathrm{d}\mu_R} \frac{\partial}{\partial \lambda} \tilde{G}_R(\mu_R, \lambda) + \\
+ \frac{n}{2} \mu_R \left(\frac{\mathrm{d} \log Z_\phi}{\mathrm{d}\mu_R} \right) \tilde{G}_R(\mu_R, \lambda(\mu_R)) = 0. \quad (7.3.9)
\end{aligned}
$$

The scale-dependence of λ and Z_ϕ appear through what we call **beta function**:

$$\beta(\lambda) = \mu_R \frac{\mathrm{d}\lambda(\mu_R)}{\mathrm{d}\mu_R} \tag{7.3.10}$$

which tells us how "fast" the coupling change with the scale; and the **anomalous dimension**:

$$\gamma_\phi(\lambda) = \frac{1}{2} \mu_R \frac{\mathrm{d} \log Z_\phi}{\mathrm{d}\mu_R} \tag{7.3.11}$$

which is the analogous of the $\beta(\lambda)$, but for the field strength. We can see that $\gamma_\phi(\lambda)$ can be written as:

$$
\begin{aligned}
\gamma_\phi(\lambda) &= \frac{1}{2} \mu_R \frac{\mathrm{d} \log Z_\phi}{\mathrm{d}\mu_R} && (7.3.12)\\
&= \frac{1}{2} \mu_R \frac{\mathrm{d}\lambda}{\mathrm{d}\mu_R} \frac{\mathrm{d}}{\mathrm{d}\lambda} \log Z_\phi && (7.3.13)\\
&= \frac{1}{2} \beta(\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda} \log Z_\phi. && (7.3.14)
\end{aligned}
$$

We have arrived at the **Callan-Symanzik equation**:

$$\left(\mu_R \frac{\partial}{\partial \mu_R} + \beta(\lambda) \frac{\partial}{\partial \lambda} + n \gamma_\phi(\lambda) \right) \tilde{G}_R = 0. \tag{7.3.15}$$

The parameters β and γ are the same for every n, and must be independent of the p_i. Since the Green's function $\tilde{G}$ is renormalized, β and γ cannot depend on the scale μ_R. Therefore they are functions only of the dimensionless variable λ. We conclude that any Green's function of massless scalar theory must satisfy (7.3.15). The Callan-Symanzik equation asserts that there exist two universal functions $\beta(\lambda)$ and $\gamma(\lambda)$, related to the shifts in the coupling constant and field strength, that compensate for the shift in the renormalization scale μ_R.

Note. This is the simplest case of massless scalar field theory. There will be additional terms for masses and other fields. Indeed, in theories with multiple fields and couplings, there is a γ term for each field and a β term for each coupling. For example, for *massive scalar theory* we have:

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}+\gamma_m(\lambda)m\frac{\partial}{\partial m}+n\gamma_\phi(\lambda)\right)\tilde{G}(\mu,\lambda(\mu),m(\mu))=0. \quad (7.3.16)$$

For *QED*, where there are more fields, we have:

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(e)\frac{\partial}{\partial e}+\gamma_m(e)m\frac{\partial}{\partial m}+n\gamma_A(e)+\right.$$
$$\left.+2k\gamma_\psi(e)\right)\tilde{G}_{n,2k}(\mu,e(\mu),m(\mu))=0. \quad (7.3.17)$$

7.4 Extraction of β and γ from UV counterterms

For a *minimal scheme* there is a direct connection between the perturbative expansion of conunterterms, $1-Z_x$, and β/γ. For example we can write:

$$\lambda_0=\lambda\mu^\epsilon Z_\lambda \quad , \quad Z_\lambda=1+\delta_\lambda \quad (7.4.1)$$

where the perturbation expansion of δ_λ is:

$$\delta_\lambda=\sum_{L=1}^{\infty}\lambda^{2L}\delta_\lambda^{(L)} \quad (7.4.2)$$

but $\delta_\lambda^{(L)}$ also has a series expansion in dimensional regularization:

$$\delta_\lambda^{(L)}=\sum_{k=-L}^{\infty}\epsilon^k\delta_\lambda^{(L,k)} \quad (7.4.3)$$

and, in a minimal scheme, the upper bound of the ϵ expansion will be -1 (only poles). Hence:

$$\delta_\lambda = \sum_{L=1}^{\infty}\sum_{k=-L}^{-1} \lambda^{2L}\epsilon^k \delta_\lambda^{(L,k)} \tag{7.4.4}$$

re-label with the values for k

$$= \sum_{L=1}^{\infty}\sum_{k=1}^{L} \frac{\lambda^{2L}}{\epsilon^k} \delta_\lambda^{(L,k)} \tag{7.4.5}$$

change order of the summation

$$= \sum_{k=1}^{\infty}\sum_{L=k}^{\infty} \frac{\lambda^{2L}}{\epsilon^k} \delta_\lambda^{(L,k)} \tag{7.4.6}$$

$$= \sum_{k=1}^{\infty} \frac{a_k(\lambda)}{\epsilon^k} \tag{7.4.7}$$

where we use:

$$a_k(\lambda) = \sum_{L=k}^{\infty} \lambda^{2L}\delta_\lambda^{(L,k)}. \tag{7.4.8}$$

So, with the definition (7.4.8), we have the expression:

$$Z_\lambda = 1 + \sum_{k=1}^{\infty} \frac{a_k(\lambda)}{\epsilon^k}. \tag{7.4.9}$$

Now, we can look at the ϵ and λ dependence of β. With the definition (7.3.10) we have:

$$\beta(\lambda,\epsilon) = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\lambda \tag{7.4.10}$$

$$= \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\lambda_0\mu^{-\epsilon}Z_\lambda^{-1}\right) \tag{7.4.11}$$

$$= -\epsilon\lambda_0\mu^{-\epsilon}Z_\lambda^{-1} + \lambda_0\mu^{-\epsilon}\mu\frac{\mathrm{d}Z_\lambda^{-1}}{\mathrm{d}\mu} \tag{7.4.12}$$

$$= -\epsilon\lambda + \lambda Z_\lambda\mu\frac{\mathrm{d}Z_\lambda^{-1}}{\mathrm{d}Z_\lambda}\frac{\mathrm{d}Z_\lambda}{\mathrm{d}\mu} \tag{7.4.13}$$

$$= -\epsilon\lambda - \lambda Z_\lambda^{-1}\beta(\lambda,\epsilon)\frac{\mathrm{d}Z_\lambda}{\mathrm{d}\lambda} \tag{7.4.14}$$

so we get:

$$\beta(\lambda,\epsilon)\left(Z_\lambda + \lambda\frac{\mathrm{d}Z_\lambda}{\mathrm{d}\lambda}\right) + \epsilon\lambda Z_\lambda = 0. \tag{7.4.15}$$

The ϵ expansion of β is:

$$\beta(\lambda,\epsilon) = \sum_{k=0}^{\infty}\epsilon^k\beta^{(k)}(\lambda) \tag{7.4.16}$$

and we are interested in terms up to $\mathcal{O}(\epsilon^0)$, so we can expand (7.4.15) up to order $N > 0$:

$$0 = \left(\beta^{(0)} + \epsilon\beta^{(1)} + \cdots + \epsilon^N \beta^{(N)}\right) \times$$
$$\times \left(1 + \frac{a_1}{\epsilon} + \frac{a_2}{\epsilon} + \cdots + \right. \tag{7.4.17}$$
$$\left. + \frac{1}{\epsilon}\lambda\frac{\mathrm{d}a_1}{\mathrm{d}\lambda} + \frac{1}{\epsilon^2}\lambda\frac{\mathrm{d}a_2}{\mathrm{d}\lambda} + \dots\right) + \tag{7.4.18}$$
$$+ \epsilon\lambda\left(1 + \frac{a_1}{\epsilon} + \frac{a_2}{\epsilon^2} + \dots\right) \tag{7.4.19}$$

where in (7.4.17), (7.4.18) and (7.4.19) there are the expansion respectively of Z_λ, $\lambda\frac{\mathrm{d}Z_\lambda}{\mathrm{d}\lambda}$ and $\epsilon\lambda Z_\lambda$. The only term at $\mathcal{O}(\epsilon^N)$ is $\beta^{(N)}$ so:

$$\mathcal{O}(\epsilon^N) \implies \beta^{(N)}\epsilon^N \tag{7.4.20}$$

but with $\beta^{(N)} = 0$ we have the lower order that gives us:

$$\mathcal{O}(\epsilon^{N-1}) \implies \beta^{(N-1)} = 0 \tag{7.4.21}$$

for $N > 1$. At $\mathcal{O}(\epsilon)$ we see the last contribution $\epsilon\lambda Z_\lambda$:

$$\mathcal{O}(\epsilon^N) \implies \beta^{(N)} = 0 \tag{7.4.22}$$
$$\mathcal{O}(\epsilon^{N-1}) \implies \beta^{(N-1)} = 0 \tag{7.4.23}$$
$$\vdots$$
$$\mathcal{O}(\epsilon) \implies \beta^{(1)} = -\lambda \tag{7.4.24}$$
$$\mathcal{O}(\epsilon^0) \implies \beta^{(0)} = \lambda^2\frac{\mathrm{d}}{\mathrm{d}\lambda}a_1 \tag{7.4.25}$$
$$\mathcal{O}(\epsilon^{-1}) \implies \beta^{(0)}\frac{\mathrm{d}(\lambda a_1)}{\mathrm{d}\lambda} = \lambda^2\frac{\mathrm{d}}{\mathrm{d}\lambda}a_2 \tag{7.4.26}$$
$$\vdots$$
$$\mathcal{O}(\epsilon^{-k}) \implies \beta^{(0)}\frac{\mathrm{d}(\lambda a_k)}{\mathrm{d}\lambda} = \lambda^2\frac{\mathrm{d}}{\mathrm{d}\lambda}a_{k+1} \tag{7.4.27}$$

where in (7.4.25) we use the fact that for $\mathcal{O}(\epsilon^0)$ we have:

$$0 = \epsilon\beta^{(1)}\frac{a_1}{\epsilon} + \beta^{(0)} + \epsilon\beta^{(1)}\frac{\lambda}{\epsilon}\frac{\mathrm{d}a_1}{\mathrm{d}\lambda} + \epsilon\lambda\frac{a_1}{\epsilon} \tag{7.4.28}$$
$$\underset{\beta^{(1)}=-\lambda}{=} \beta^{(0)} - \lambda^2\frac{\mathrm{d}a_1}{\mathrm{d}\lambda} \tag{7.4.29}$$

and in (7.4.26):

$$\beta^{(0)}\frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda a_1) = \lambda\frac{\mathrm{d}a_1}{\mathrm{d}\lambda} + a_1. \tag{7.4.30}$$

In particular this relation shows us:

$$\lambda \frac{\mathrm{d}a_1}{\mathrm{d}\lambda} + a_1 = \lambda^2 \frac{\mathrm{d}}{\mathrm{d}\lambda} a_2 \tag{7.4.31}$$

and in facts, you can see in Nastase Let no one ignorant of geometry enter my door.Nastase the derivation, there is a recursion relation between the coefficients of poles at order $\mathcal{O}(\epsilon^{-k})$:

$$\left(\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} - 1\right) a_{k+1}(\lambda) = \beta(\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda} a_k(\lambda) \tag{7.4.32}$$

and this means that we can determine (in principle) all the $a_k(\lambda)$ in terms of $a_1(\lambda)$ only.

So $\beta^{(0)} = \beta(\lambda, 0)$ is the beta function in the CS equation and appears in multiple relations, so we find relations between the poles of δ_λ, recalling (7.4.8), at different orders in ϵ and L:

$$\beta^{(0)} = \lambda^2 \frac{\mathrm{d}}{\mathrm{d}\lambda} a_1 \tag{7.4.33}$$

$$\beta^{(0)} \frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda a_1) = \lambda^2 \frac{\mathrm{d}}{\mathrm{d}\lambda} a_2 \tag{7.4.34}$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}\lambda} a_1 \frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda a_1) = \frac{\mathrm{d}}{\mathrm{d}\lambda} a_2 \tag{7.4.35}$$

$$\overset{\mathcal{O}(\lambda^3)}{\implies} \left(\delta_\lambda^{(1,1)}\right)^2 = \frac{2}{3} \delta_\lambda^{(2,2)}. \tag{7.4.36}$$

In summary we find:

$$\beta(\lambda) = \lambda^2 \frac{\mathrm{d}}{\mathrm{d}\lambda} a_1(\lambda) \tag{7.4.37}$$

where is valid (7.4.8) and (7.4.9). This formula is exact to all orders in perturbation theory, and note that we only need to know the coefficient of the single pole, $a_1(\lambda)$, not of any of the higher-order poles.

A similar relation holds for γ_ϕ:

$$\gamma_\phi = -\frac{1}{2} \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} c_1(\lambda) \tag{7.4.38}$$

where:

$$Z_\phi = 1 + \sum_{k=1}^{\infty} \frac{c_k(\lambda)}{\epsilon^k}. \tag{7.4.39}$$

7.4.1 Calculation of β and γ in $\phi^3_{d=6}$

Recall the values obtained in Caphter §3 for the $\overline{MS}$ scheme (here we convert to MS):

$$\delta_1 = -\frac{\lambda^2}{(4\pi)^3}\frac{1}{2\epsilon} + \mathcal{O}(\lambda^4) \tag{7.4.40}$$

$$= \lambda^2 \delta_1^{(1,1)}\frac{1}{\epsilon} + \mathcal{O}(\lambda^4) \tag{7.4.41}$$

$$\delta_\phi = -\frac{\lambda^2}{(4\pi)^3}\frac{1}{12\epsilon} + \mathcal{O}(\lambda^4) \tag{7.4.42}$$

$$= \lambda^2 \delta_\phi^{(1,1)}\frac{1}{\epsilon} + \mathcal{O}(\lambda^4) \tag{7.4.43}$$

so we find:

$$\delta_\lambda^{(1,1)} = \delta_1^{(1,1)} - \frac{3}{2}\delta_\phi^{(1,1)} = -\frac{1}{(4\pi)^3}\frac{3}{8} \tag{7.4.44}$$

and the beta function:

$$\beta(\lambda) = \lambda^2 \frac{\mathrm{d}}{\mathrm{d}\lambda} a_1(\lambda) \tag{7.4.45}$$

$$= \lambda^2 \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\lambda^2 \delta^{(1,1)} + \mathcal{O}(\lambda^4)\right) \tag{7.4.46}$$

$$= 2\lambda^3 \delta^{(1,1)} + \mathcal{O}(\lambda^5) \tag{7.4.47}$$

$$= -\frac{3}{4}\frac{\lambda^3}{(4\pi)^3} + \mathcal{O}(\lambda^5). \tag{7.4.48}$$

Exercise 15. Check that:

$$\gamma_\phi(\lambda) = \frac{1}{12}\frac{\lambda^2}{(4\pi)^3} + \mathcal{O}(\lambda^4). \tag{7.4.49}$$

Observation. Since $\beta(\lambda) < 0$ for small λ we have:

$$\mu\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = \frac{\mathrm{d}\lambda}{\mathrm{d}\log(\mu)} < 0 \tag{7.4.50}$$

so, $\lambda(\mu)$ decreases as μ increases. See plot in figure 7.1.

7.4.2 Calculation in QED

We have:

$$\delta_e = \delta_1 - \delta_\psi - \frac{1}{2}\delta_A \tag{7.4.51}$$

$$= -\frac{1}{2}\delta_A \tag{7.4.52}$$

$$= \left(\frac{\alpha}{4\pi}\right)\frac{2}{3\epsilon} + \mathcal{O}(\alpha^2) \tag{7.4.53}$$

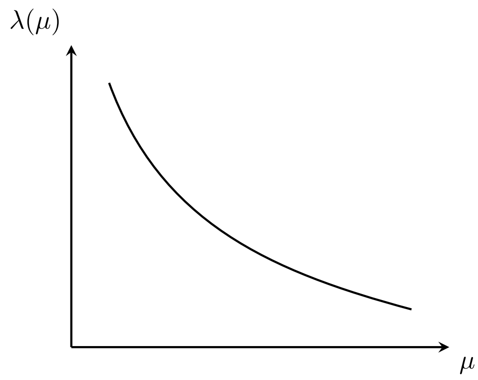


Figure 7.1

and since:

$$\alpha = \frac{e^2}{4\pi} \tag{7.4.54}$$

we get:

$$\delta_e = \frac{e^2}{16\pi^2}\frac{2}{3\epsilon} + \mathcal{O}(e^4). \tag{7.4.55}$$

The beta function, dependent on e, is:

$$\beta(e) = \frac{e^3}{16\pi^2}\frac{4}{3} + \mathcal{O}(e^5) \tag{7.4.56}$$

$$= e\left[\left(\frac{\alpha}{4\pi}\right)\frac{4}{3} + \mathcal{O}(\alpha^2)\right]. \tag{7.4.57}$$

NB. We can see that $\beta(e) \neq \beta(\alpha)$, and we have:

$$\beta(\alpha) = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\alpha(\mu) \tag{7.4.58}$$

$$= \frac{1}{4\pi}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}e^2(\mu) \tag{7.4.59}$$

$$= \frac{e}{2\pi}\beta(e) \tag{7.4.60}$$

$$= 2\alpha\left[\left(\frac{\alpha}{4\pi}\right)\frac{4}{3} + \mathcal{O}(\alpha^2)\right]. \tag{7.4.61}$$

Although we can see we have $\beta(\alpha) > 0$ for small α, and that implies:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\log\mu} > 0 \tag{7.4.62}$$

so the coupling increases with the scale. As you can see in figure 7.2.

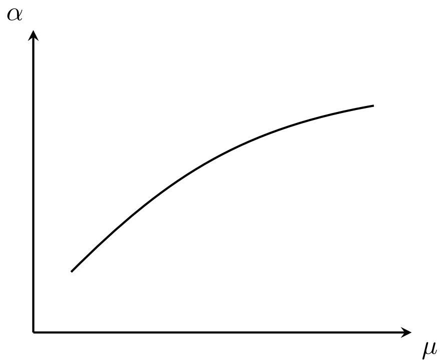


Figure 7.2

Exercise 16. Show the anomolous dimension are:

$$\gamma_\psi(e) = -\frac{1}{2}e\frac{\mathrm{d}c_1}{\mathrm{d}e} = \frac{e^2}{16\pi^2} + \mathcal{O}(e^4) \tag{7.4.63}$$

$$\gamma_A(e) = \frac{4}{3}\frac{e^2}{16\pi^2} + \mathcal{O}(e^4). \tag{7.4.64}$$

7.4.3 Calculation in QCD

For QCD we've found in section §6.2[1]:

$$\delta_{\psi_q} = \left(\frac{\alpha_s}{4\pi}\right)\frac{1}{\epsilon}(-C_F) + \mathcal{O}(\alpha_s^2) \tag{7.4.65}$$

$$\delta_A = \left(\frac{\alpha_s}{4\pi}\right)\frac{1}{\epsilon}(5C_A - 4T_R n_f)\frac{1}{3} + \mathcal{O}(\alpha_s^2) \tag{7.4.66}$$

$$\delta_1 = \left(\frac{\alpha_s}{4\pi}\right)\frac{1}{\epsilon}(-C_A - C_F) + \mathcal{O}(\alpha_s^2) \tag{7.4.67}$$

that implies:

$$\delta_{g_s} = \left(\frac{\alpha_s}{4\pi}\right)\frac{1}{\epsilon}(-11C_A + 4T_R n_f)\frac{1}{6} + \mathcal{O}(\alpha_s^2) \tag{7.4.68}$$

and:

$$\beta(g_s) = -g_s\left(\frac{\alpha_s}{4\pi}\right)\left(\frac{11C_A - 4T_R n_f}{3}\right) + \mathcal{O}(g_s^5) \tag{7.4.69}$$

$$\beta(\alpha_s) = -\alpha_s\left(\frac{\alpha_s}{4\pi}\right)\frac{2}{3}(11C_A - 4T_R b_f) + \mathcal{O}(\alpha_s^5) \tag{7.4.70}$$

$$= -\alpha_s\left[\left(\frac{\alpha_s}{2\pi}\right)b_0 + \mathcal{O}(\alpha_s^2)\right] \tag{7.4.71}$$

[1]Noticing that the coefficient C_Γ in the expression of δ gives contribution 1 to the counterterms at first order.

where we defined:

$$b_0 = \frac{1}{3}(11C_A - 4T_R n_f). \tag{7.4.72}$$

Observation. As we could notice for the $\phi^3_{d=6}$ theory, for small α_s and for $n_f \leq 16$ we have:

$$\frac{\mathrm{d}\alpha_s}{\mathrm{d}\log\mu} < 0 \tag{7.4.73}$$

so the coupling decreases with the scale, and we have what is called *asymptotic freedom*, which indicates we may apply perturbation theory at high energies (but not at low energies). See the figure 7.3.

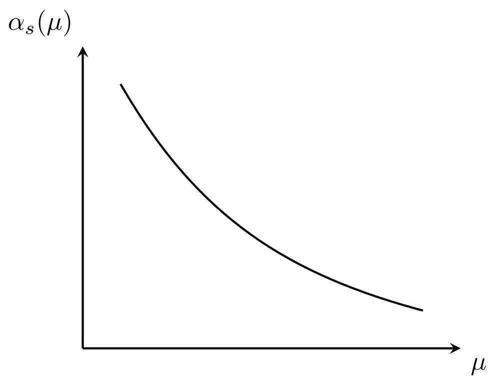


Figure 7.3

7.5 Solution to the Callan-Symanzik equation

Keep in mind that we have in our hand a *conformal theory*, where its scale invariance (classical symmetry breaks by Quantum Mechanics) is breaks by the renormalization process, which introduces a mass scale. We won't gain any problematic behaviors because the renormalization group's equation will force the correspondence between the physical scale and the renormalization mass scale.

We start by reminding ourselves of the form of the 2-point correlation function in a scalar theory (massless):

$$\tilde{G}_2(p^2; \mu, \lambda(\mu)) = \frac{i}{p^2 + i\emptyset^+} g_2(-\mu^2/p^2, \lambda(\mu)) \tag{7.5.1}$$

where:

$$g_s(-\mu^2/p^2, \lambda(\mu)) = 1 + \lambda^2 g_2^{(1)}(-\mu^2/p^2) + \mathcal{O}(\lambda^4). \tag{7.5.2}$$

We wrote $\tilde{G}_2$ as (7.5.1) because we have $\tilde{G}_2$ with dimension of $(\text{mass})^{-2}$.

Since g_2 depends only on the ratio $-\mu^2/p^2$, we can swap $\mu\frac{\partial}{\partial\mu}$ for $p^\mu\frac{\partial}{\partial p^\mu}$ in the CS equation (7.3.15):

$$\left(p^\mu\frac{\partial}{\partial p^\mu} - \beta(\lambda)\frac{\partial}{\partial\lambda} + 2(1-\gamma_\phi(\lambda))\right)\tilde{G}_2 = 0. \tag{7.5.3}$$

Proof. We can start noticing:

$$\mu\frac{\partial}{\partial\mu} = 2\mu^2\frac{\partial}{\partial\mu^2} \quad , \quad p^\mu\frac{\partial}{\partial p^\mu} = 2p^2\frac{\partial}{\partial p^2} \tag{7.5.4}$$

let $x = -\mu^2/p^2$ so that:

$$\mu^2\frac{\partial}{\partial\mu^2}g_2(x,\lambda) = \mu^2\frac{\partial x}{\partial\mu^2}\frac{\partial}{\partial x}g_2(x,\lambda) \tag{7.5.5}$$

$$= x\frac{\partial}{\partial x}g_2(x,\lambda) \tag{7.5.6}$$

$$p^2\frac{\partial}{\partial\mu^2}g_2(x,\lambda) = -x\frac{\partial}{\partial x}g_2(x,\lambda) \tag{7.5.7}$$

from which we show:

$$\mu\frac{\partial}{\partial\mu}\tilde{G}_2 = -\left(p^\mu\frac{\partial}{\partial p^\mu} + 2\right)\tilde{G}_2 \tag{7.5.8}$$

leading to the desired form of the CS equation. ■

7.5.1 Solution 1: $\beta = \gamma_\phi = 0$ - Free theory

We start with the free theroy, so $\beta = \gamma_\phi = 0$, and we want to solve the Callan-Symanzik's equation for the two-point Green's function, in a theory with a single scalar field:

$$\left(p^\mu\frac{\partial}{\partial p^\mu} + 2\right)\tilde{G}_2 = 0 \tag{7.5.9}$$

$$\implies \left(p^2\frac{\partial}{\partial p^2} + 1\right)\tilde{G}_2 = 0 \tag{7.5.10}$$

$$\implies \tilde{G}_2 = \frac{k}{p^2} \;, \quad k = \text{constant}. \tag{7.5.11}$$

We can rewrite (7.5.10) using the mass dimension of the scalar field $\Delta = 1$:

$$p^2\frac{\partial}{\partial p^2}\tilde{G}_2 = -\Delta\tilde{G}_2 \tag{7.5.12}$$

if we release the condition $\gamma_\phi = 0$, the CS equation is:

$$\frac{\partial}{\partial\log p^2}\tilde{G}_2 = -(1-\gamma_\phi)\tilde{G}_2 \tag{7.5.13}$$

which demonstrates why γ_ϕ is called the *anomalous dimension*:

$$\Delta = 1-\gamma_\phi \quad , \quad \tilde{G}_2 = \frac{k}{(p^2)^\Delta}. \tag{7.5.14}$$

7.5.2 Solution 2: Coleman's Hydrodynamic/Bacterialogical analogy

We can talk about *Coleman's Hydrodynamic/Bacterialogical analogy.* See chapter §12, p. 418, in Peskin Let no one ignorant of geometry enter my door.Peskin for detail.

We consider a fluid flowing with velocity $v(x)$ along the axis of a narrow tube. As in figure 7.4.

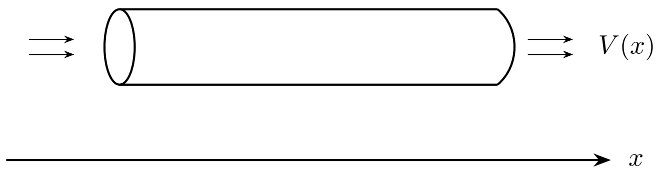


Figure 7.4

The liquid contains bacteria distributed according to a function $D(t, x)$ at time t and position x. The tube is illumated by light distributed according to a function $\rho(x)$. As you can see in figure 7.5.

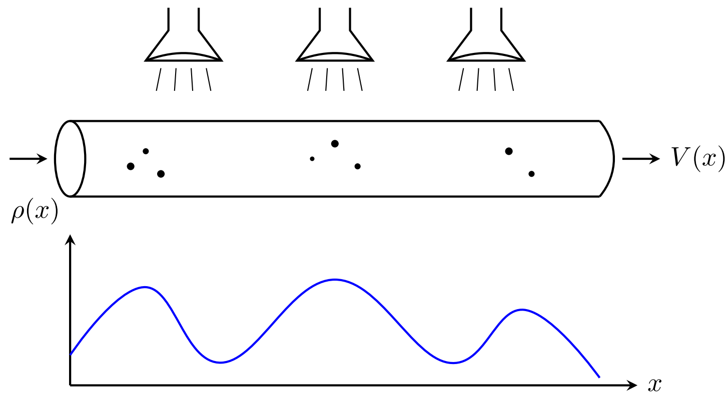


Figure 7.5: Coleman's bacteriological analogy to the Callan-Symanzik equation. The pipe is inhabited by bacteria with a given initial density $D_{init}(x)$. The growth rate (determined by the illumination) and flow velocity are given functions of x. The problem is to determine the density $D(t, x)$ at all subsequent times.

The bacteria density will follow an evolution equation almost identical to the CS equation for correlation functions:

$$\left(\frac{\partial}{\partial t} + v(x)\frac{\partial}{\partial x} - \rho(x)\right) D(t, x) = 0. \tag{7.5.15}$$

The translation can be summarized as in table 7.1.

Tube model	CS equation
$\frac{\partial}{\partial t}$	$\frac{\partial}{\partial \log(\mu/\lvert p\rvert)}$
x	λ
$v(x)$	$-\beta(\lambda)$
$\rho(x)$	$2(\gamma(\lambda)-1)$
$D(t,x)$	$g_2(-p^2/\mu^2, \lambda(\mu))$

Table 7.1

We will find solutions according to the initial condition:

$$D(0,x) = D_{init}(x). \tag{7.5.16}$$

Let's solve some special cases first:

Static case: $v(x) = 0$, so we have:

$$\left(\frac{\partial}{\partial t} - \rho(x)\right) D(t,x) = 0 \tag{7.5.17}$$

which have solution:

$$D(t,x) = \exp\{\rho(x)t\} D_{init}(x). \tag{7.5.18}$$

No illumination: $\rho(x) = 0$, so we have:

$$\left(\frac{\partial}{\partial t} + v(x)\frac{\partial}{\partial x}\right) D(t,x) = 0 \tag{7.5.19}$$

in the case $v(x) = v$ is a constant, we find a simple travelling wave solution:

$$D(t,x) = D(x - vt) \quad , \qquad z = x - vt \tag{7.5.20}$$

$$\frac{\partial}{\partial t} D(z) = -vD'(z) \quad , \qquad \frac{\partial}{\partial x} D(x) = D'(z) \tag{7.5.21}$$

so:

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) D(z) = 0. \tag{7.5.22}$$

For the general[2] case we may write the solution using a fluid element which moves according to $\hat{x}(t)$. At a time $t = 0$ we find the element at x_0:

$$\hat{x}(0) = x_0. \tag{7.5.23}$$

Since the β function is $-v$ in the analogy we define $\hat{x}(t)$ to be the position for $t < 0$. The fluid element is described by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t) = -v(\hat{x}(t)) \ , \quad \hat{x}(0) = x_0 \tag{7.5.24}$$

[2]From now, untill the end of this case, we will make a very similar calculation as what we've done in *Phenomenology of Fundamental Interaction* MSc course, so it could be useful to see the dedicated section of running coupling in those notes.

which has the solution:

$$\int_{x_0}^{\hat{x}(t)} \frac{\mathrm{d}\hat{x}}{v(\hat{x})} = -\int_0^t \mathrm{d}t' = -t. \tag{7.5.25}$$

The dependence on the initial condition can be denoted $\hat{x} \equiv \hat{x}(t; x_0)$. Let's change the initial condition to be somewhere arbitrary $x_0 \to x$ and find $\hat{x}(t; x)$ that satisfies both:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t; x) = -v(\hat{x}(t, x)) \tag{7.5.26}$$

$$\left(\frac{\partial}{\partial t} + v(x)\frac{\partial}{\partial x}\right)\hat{x}(t; x) = 0. \tag{7.5.27}$$

To clarify we introduce a primitive:

$$P = \frac{\mathrm{d}\hat{x}}{v(\hat{x})} \tag{7.5.28}$$

with inverse U such that:

$$U(P(y)) = y \tag{7.5.29}$$

and:

$$-t = \int_x^{\hat{x}(t;x)} \frac{\mathrm{d}\hat{x}}{v(\hat{x})} \tag{7.5.30}$$

$$= \int_x^{\hat{x}(t;x)} \mathrm{d}P \tag{7.5.31}$$

$$= P(\hat{x}(t; x)) - P(x) \tag{7.5.32}$$

so we get:

$$\hat{x}(t; x) = U(P(x) - t) \tag{7.5.33}$$

which solves (7.5.26).

Now check (7.5.27):

$$\frac{\partial \hat{x}}{\partial t} = \frac{\partial}{\partial t}(P(x) - t)U'(P(x) - t) \tag{7.5.34}$$

$$= -U'(P(x) - t) \tag{7.5.35}$$

$$\frac{\partial \hat{x}}{\partial x} = \frac{\partial}{\partial x}(P(x) - t)U'(P(x) - t) \tag{7.5.36}$$

$$= \frac{\mathrm{d}P(x)}{\mathrm{d}x}U'(P(x) - t) \tag{7.5.37}$$

$$= \frac{1}{v(x)}U'(P(x) - t) \tag{7.5.38}$$

$$= \frac{1}{v(x)}\left(-\frac{\partial \hat{x}}{\partial t}\right) \tag{7.5.39}$$

which confirms equations (7.5.27) is satisfied. ■

Now we can build the solution for $D(x,t)$ using $\hat{x}(t;x)$:

$$D(x,t) = D_{init}(\hat{x}(t;x)) \tag{7.5.40}$$

where:

$$\begin{aligned} D_{init}(x) &= D_{init}(\hat{x}(0;x)) && (7.5.41) \\ &= D(0,x). && (7.5.42) \end{aligned}$$

We can verify the solution. We see:

$$\begin{aligned} 0 &= \left(\frac{\partial}{\partial t} + v(x)\frac{\partial}{\partial x}\right) D(t,x) && (7.5.43) \\ &= \frac{\partial}{\partial t} D_{init}(\hat{x}(t;x)) + v(x)\frac{\partial}{\partial x} D_{init}(\hat{x}(t;x)) && (7.5.44) \\ &= \frac{\partial \hat{x}}{\partial t} D'_{init} + v(x)\frac{\partial \hat{x}}{\partial x} D'_{init} && (7.5.45) \\ &= D'_{init}\left(\frac{\partial \hat{x}}{\partial t} + v(x)\frac{\partial \hat{x}}{\partial x}\right) && (7.5.46) \\ &\overset{(7.5.27)}{=} 0. && (7.5.47) \end{aligned}$$

So we've found that (7.5.40) is our solution. ■

General case: let's see the case with $\rho(x) \neq 0$ and $v(x) \neq 0$. The solution can be found with the *characteristic method*. The general case is a combination of the solutions found already, indeed:

$$\begin{aligned} D(t,x) &= D_{init}(\hat{x}(t;x)) \exp\left\{\int_0^t \mathrm{d}t'\, \rho(\hat{x}(t;x))\right\} && (7.5.48) \\ &\equiv f_1(t,x) f_2(t,x). && (7.5.49) \end{aligned}$$

The solution is straightforward to verify, calling:

$$O(t,x) = \frac{\partial}{\partial t} + v(x)\frac{\partial}{\partial x} \tag{7.5.50}$$

we have:

$$\begin{aligned} 0 &= \Big(O(t,x) - \rho(x)\Big) D(t,x) && (7.5.51) \\ &= \underbrace{O(f_1)}_{=0} f_2 + f_1 O(f_2) - \rho f_1 f_2 && (7.5.52) \\ &= f_1 \left[\rho(\hat{x}) f_2 + \left(\int_0^t \mathrm{d}t'\, v(x)\frac{\partial}{\partial x}\rho(\hat{x})\right) f_2\right] - f_1 f_2 \rho(x) && (7.5.53) \\ &= f_1 f_2 \left(\rho(\hat{x}) - \rho(x) + \int_0^t \mathrm{d}t'\, v(x)\frac{\partial}{\partial x}\rho(\hat{x})\right). && (7.5.54) \end{aligned}$$

Now, recall that:

$$O(t,x)\hat{x}(t;x) = 0 \tag{7.5.55}$$

$$\implies \quad v(x)\frac{\partial}{\partial x}\rho(\hat{x}) = -\frac{\partial}{\partial t}\rho(\hat{x}) \tag{7.5.56}$$

$$\implies \quad \int_0^t \mathrm{d}t'\, v(x)\frac{\partial}{\partial x}\rho(\hat{x}(t';x)) = -\int_0^t \mathrm{d}t'\, \frac{\partial}{\partial t'}\rho(\hat{x}(t';x)) \tag{7.5.57}$$

$$= -\rho(\hat{x}(t;x)) + \rho(\hat{x}(0;x)) \tag{7.5.58}$$

$$= -\rho(\hat{x}) + \rho(x) \tag{7.5.59}$$

which completes the validation of our solution. ■

NB. An alternative form of the solution is:

$$D(t,x) = D_{init}(\hat{x}(t;x))\exp\left\{-\int_x^{\hat{x}(t;x)} \mathrm{d}x'\, \frac{\rho(x')}{v(x')}\right\}. \tag{7.5.60}$$

Now we can translate, using the table 7.1, the solution back to the 2-point correlation function for our field theory, and that leads to:

$$\tilde{G}_2(p,\mu,\lambda(\mu)) = G_{init}(\mu,\hat{\lambda}(p;\lambda))\times$$
$$\exp\left\{-\int_{|p'|=\mu}^{|p'|=|p|} \mathrm{d}\log\left(\frac{|p'|}{\mu}\right)2(1-\gamma_\phi(\hat{\lambda}(p;\lambda)))\right\} \tag{7.5.61}$$

where:

$$\frac{\mathrm{d}}{\mathrm{d}\log\left(\frac{|p|}{\mu}\right)}\hat{\lambda}(p;\lambda) = \beta(\hat{\lambda}). \tag{7.5.62}$$

The differential equation (7.5.62) describes the flow of a modified coupling constant $\hat{\lambda}(p;\lambda)$ as a function of momentum. The rate of this flow is just the β function. We will refer to $\hat{\lambda}(p)$ as the *running coupling constant*. Its equation (7.5.62) is often called the *renormalization group equation*.

We can notice that since:

$$\exp\left\{-2\int \mathrm{d}\log\left(\frac{|p'|}{\mu}\right)\right\} = \frac{\mu^2}{p^2} \tag{7.5.63}$$

a convenient way to write the solution (7.5.61) is:

$$\tilde{G}_2(p,\mu,\lambda(\mu)) = \frac{-i}{p^2}g_2'(\mu,\hat{\lambda}(p;\lambda))\exp\left\{2\int \mathrm{d}\log\left(\frac{|p'|}{\mu}\right)\gamma_\phi(\hat{\lambda})\right\} \tag{7.5.64}$$

$$\equiv \frac{-i}{p^2}g_2(-\mu^2/p^2,\hat{\lambda}(p;\lambda)) \tag{7.5.65}$$

where $\hat{\lambda}(\mu;\lambda) = \lambda$ and g_2' is a function that must be determined. This function cannot be determined from the general principles of renormalization theory. Instead, we must compute $\tilde{G}^2(p)$ as a perturbation series in λ and match terms to the expansion of (7.5.64) as a series in the same parameter.

7.6 Running couplings

Let's explore the solutions α and α_s using the β function evaluated up to 1-loop. We saw in the section (and subsection) §7.4 that for the QCD is valid:

$$\beta(\alpha_s) = -\left(\frac{\alpha_s^2}{2\pi}\right) b_0 + \mathcal{O}(\alpha_s^3) \tag{7.6.1}$$

where:

$$b_0 = \frac{11C_A - 4T_R n_f}{3}. \tag{7.6.2}$$

So, using the equation (7.3.10) (so the analogous of (7.5.25)) we can see:

$$\Longrightarrow \quad \frac{\mathrm{d}\alpha_s}{\alpha_s^2} = -\frac{b_0}{2\pi}\frac{\mathrm{d}\mu}{\mu} \tag{7.6.3}$$

$$\Longrightarrow \quad \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{\mathrm{d}\alpha_s}{\alpha_s^2} = -\frac{b_0}{2\pi}\int_{\mu_0}^{\mu}\frac{\mathrm{d}\mu'}{\mu'} \tag{7.6.4}$$

$$\Longrightarrow \quad \left[-\frac{1}{\alpha_s}\right]_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} = -\frac{b_0}{2\pi}\log\left(\frac{\mu}{\mu_0}\right) \tag{7.6.5}$$

$$\Longrightarrow \quad -\frac{1}{\alpha(\mu)} + \frac{1}{\alpha(\mu_0)} = -\frac{b_0}{2\pi}\log\left(\frac{\mu}{\mu_0}\right) \tag{7.6.6}$$

$$\Longrightarrow \quad \alpha(\mu) = \frac{\alpha(\mu_0)}{1 + \frac{b_0}{2\pi}\log\left(\frac{\mu}{\mu_0}\right)\alpha(\mu_0)} \tag{7.6.7}$$

so we can write the coupling as:

$$\alpha(\mu) = \frac{\alpha(\mu_0)}{1 + \frac{b_0}{4\pi}\log\left(\frac{\mu^2}{\mu_0^2}\right)\alpha(\mu_0)}. \tag{7.6.8}$$

We can see from this expression that $\alpha(\mu)$ depends on another scale μ_0. Indeed, we need a experimental value $\alpha(\mu_0)$ to determine $\alpha(\mu)$ to another scale.

Starting from the expression (7.6.8) we can study the QED and QCD cases.

To study the QED evolution we have to use $T_R = 1$, $C_A = 0$ and $n_f = 1$ in the expression (7.6.2), so $b_0^{QED} = -4/3$. The $\alpha(\mu)$ plot is the figure 7.6. **NB.** Note that α in the Standard Model recives corrections from quarks a W/Z boson, when above threshold:

$$\alpha^{SM}(M_Z) \approx \frac{1}{128}. \tag{7.6.9}$$

In the same way if we want to see the QCD evolution, which have $N_c = 3$, $n_f = 5$ and $T_R = 1/2$ in (7.6.2), so $b_0 = 23/3$, we can see the figure 7.7.

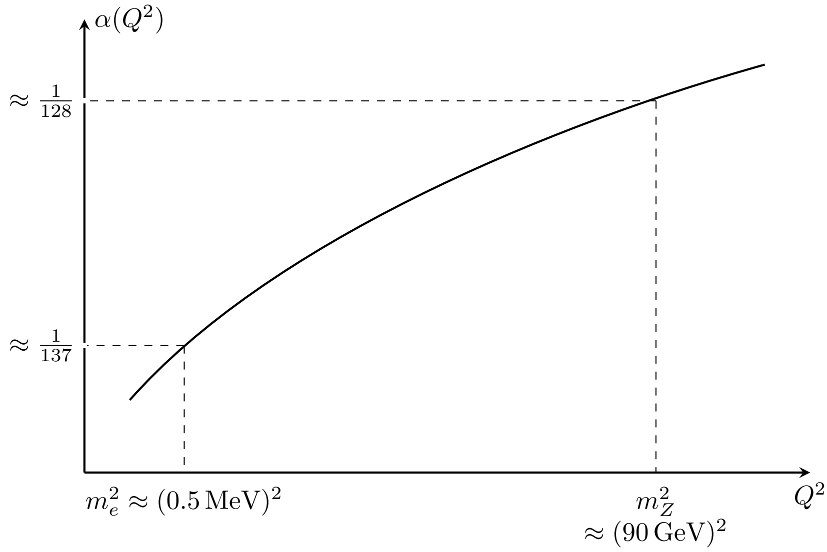


Figure 7.6

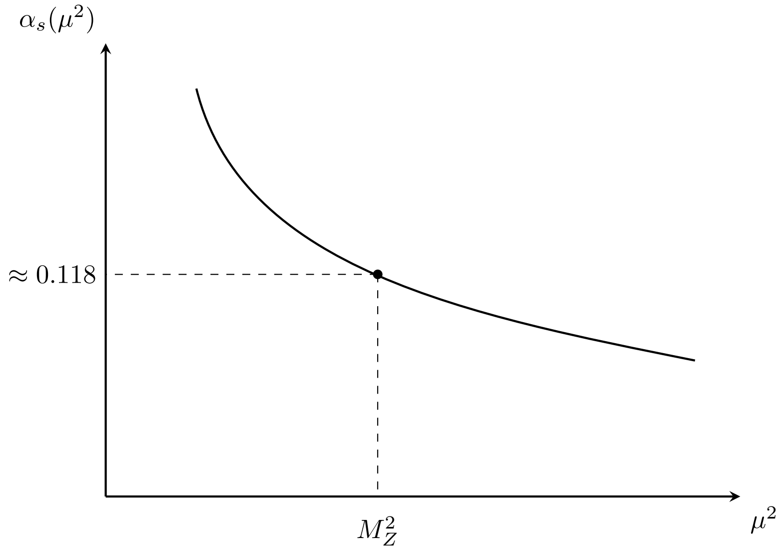


Figure 7.7

We can notice that, ignoring the different group factors N_c, T_R and the active flavour n_f, the "only" difference between QED and QCD is the sign of b_0, so the sing of $\beta(\alpha_s)$. This sign changing is crucial for the according between the experimental data and the physical behavior of abelian and non-abelian gauge theory. Before invastigate deeper this aspect let's an important property of the running coupling.

Indeed, we can notice that the running coupling (7.6.8) has a pole at:

$$1+\left(\frac{b_0}{4\pi}\right)\alpha(\mu_0)\log\left(\frac{\mu_L^2}{\mu_0^2}\right)=0 \tag{7.6.10}$$

where μ_L^2 is the location of what we call the *Landau pole*. We can see:

$$\mu_L^2=\mu_0^2\exp\left\{-\frac{4\pi}{b_0\alpha(\mu_0)}\right\} \tag{7.6.11}$$

with μ_0 a specific scale (experimentally determined) we can use to determine the value of the coupling. In QED we find:

$$\mu_L^{QED}\approx m_e\exp\{650\} \tag{7.6.12}$$

ans so way above any physical scale. For QCD the Landau pole is inside the confined region where $\alpha_s>1$:

$$\mu_L^{QCD}=M_Z\exp\left\{-\frac{2\pi}{b_0\alpha_s(M_Z)}\right\} \tag{7.6.13}$$

$$\approx 250\text{ MeV} \tag{7.6.14}$$

for $n_f=3$ active flavour.

7.6.1 Alternatives for the running of coupling constants

We refer to the chapter §12.3 from Peskin Let no one ignorant of geometry enter my door.Peskin. We've said in the last subsection that different theories has different behavior of the running coupling, and it's necessary to have a correspondence between the experiments and the mathematical model.

We are going to talk only about renormalizable theories in the massless limit, with a single dimensionless coupling constant λ.

By the arguments of the previous section, the Green's functions in any such theory obey a Callan-Symanzik equation. The solution of this equation depends on a running coupling constant, $\hat{\lambda}(p)$, which satisfies a differential equation (7.5.62) in which the function $\beta(\lambda)$ is computable as a power series in the coupling constant. In the examples we have just discussed, QED, the leading coefficient in this power series was positive, and for QCD it was negative. Indeed, as a matter of principle, three behaviors are possible in the region of small λ:

$$\beta(\lambda)>0$$
$$\beta(\lambda)=0$$
$$\beta(\lambda)<0.$$

In theories of the first class, the running coupling constant goes to zero in the infrared, leading to definite predictions about the small-momentum behavior of the theory. However, the running coupling constant becomes large in the region of high momenta. Thus the short-distance behavior of the theory cannot be computed using Feynman diagram perturbation theory. In fact, in the examples such as QED, the coupling constant formally goes to infinity at a large but finite value of the momentum. A Feynman diagram analysis is useful in such theories if one is mainly interested in large-distance or macroscopic behavior.

In theories of the second class, the coupling constant does not flow. In these theories, the running coupling constant is independent of the momentum scale, and thus equal to the bare coupling. This means that there can be no ultraviolet divergences in the relation of coupling constants. The only possible ultraviolet divergences in such theories are those associated with field rescaling, which automatically cancel in the computation of S-matrix elements. Such theories are called *finite quantum field theories*. Before the emergence of our modern understanding of renormalization, these theories would have been embraced as the solution to the problem of ultraviolet infinities. But in fact the known finite field theories in four dimensions are very special constructions —– the so-called gauge theories with extended supersymmetry —– with no known physical application.

In theories of the third class, the running coupling constant becomes large in the large-distance regime and becomes small at large momenta or short distances. This is the case of QCD. This coupling constant tends to zero at a logarithmic rate as the momentum scale increases. Such theories are called *asymptotically free*. In theories of this class, the short-distance behavior is completely solvable by Feynman diagram methods. Though ultraviolet divergences appear in every order of perturbation theory, the renormalization group tells us that the sum of these divergences is completely harmless.

7.6.2 Higher order running couplings

Analytic solution to the running coupling at higher orders are not available in general, but numerical solutions may be obtained. We may find analytic expressions when keep only the dominant logarithmic contributions. We find:

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\alpha = -\alpha\left(\frac{\alpha}{2\pi}b_0 + \left(\frac{\alpha}{2\pi}\right)^2 b_1 + \mathcal{O}(\alpha^3)\right) \tag{7.6.15}$$

$$= -\alpha^2\beta_0\left(1 + \alpha\frac{\beta_1}{\beta_0} + \mathcal{O}(\alpha^2)\right) \tag{7.6.16}$$

where:

$$\beta_k = \frac{b_k}{(2\pi)^{k+1}}. \tag{7.6.17}$$

So we find:

$$\implies \quad -\frac{\mathrm{d}\mu}{\mu} = \frac{\mathrm{d}\alpha}{\alpha^2\beta_0}\frac{1}{1+\alpha\frac{\beta_1}{\beta_0}} \tag{7.6.18}$$

$$\implies \quad -\log\left(\frac{\mu}{\mu_0}\right) = c - \frac{1}{\beta_0\alpha} - \frac{\beta_1}{\beta_0^2}\left[\log(\alpha) - \log\left(1+\alpha\frac{\beta_1}{\beta_0}\right)\right] \tag{7.6.19}$$

where c is a constant of integration. We can find the solution iteractively. First set $\beta_1 = 0$ and choose c such that:

$$-\log\left(\frac{\mu}{\Lambda}\right) = -\frac{1}{\alpha\beta_0} \equiv -L \quad , \quad c = \log\left(\frac{\mu_0}{\Lambda}\right). \tag{7.6.20}$$

In other words:

$$\alpha = \frac{1}{\beta_0 L}. \tag{7.6.21}$$

Now we take the leading log limit of the two loop equation, correcting the integration constant:

$$c = \log\left(\frac{\mu_0}{\Lambda}\right) - 2\frac{\beta_1}{\beta_0^2}\log(\beta_0) \tag{7.6.22}$$

which leads to (substituting $\alpha = 1/(\beta_0 L)$ into the argument of log then expand):

$$-L = -\frac{1}{\alpha\beta_0} + \frac{\beta_1}{\beta_0^2}\log L \tag{7.6.23}$$

so we find:

$$\alpha = \frac{1}{\beta_0 L}\left(1 - \frac{\beta_1}{\beta_0}\frac{\log L}{L}\right) + \mathcal{O}\left(\frac{1}{L^3}\right). \tag{7.6.24}$$

In QCD one finds that:

$$b_1 = \frac{34}{3}C_A^2 - \frac{20}{3}C_A T_R n_f - 4C_F T_R n_f \tag{7.6.25}$$

and the effect of $2L$ to IL running can be $1-2\%$ (for modern collider energy scales).

7.6.3 All order behaviour of β

For the limit $\alpha \ll 1$ we can see the plot in figure 7.8, where we can stop the β calculation at the first order.

But, we can ask yourself what happens for larger values of α. All of our explicit solutions for running coupling constants predict that the running coupling becomes infinite at a finite value of the momentum. It is possible that this is the true behavior of the quantum field theory, but we have not proved this, because when the running coupling constant becomes large, the approximation we have made, ignoring the higher-order terms in the β

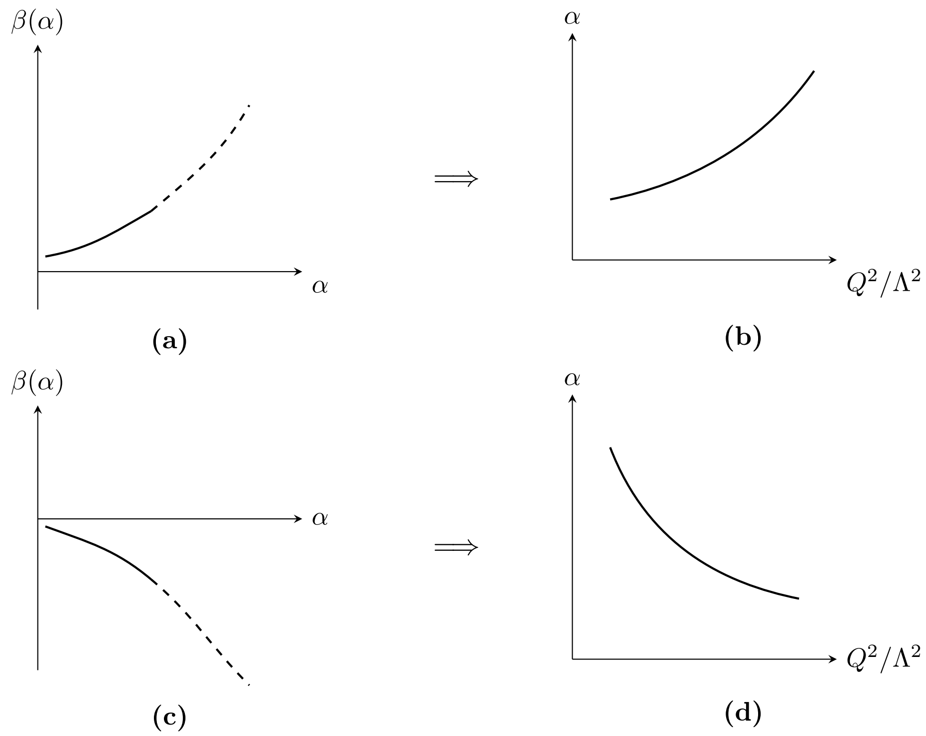


Figure 7.8

function, is no longer valid. It is a logical possibility that the higher terms of the β function are negative, so that the β function has the form shown in figure 7.9b. In this case the β function has a zero at a nonzero value α_*.

What we are saying is that it is possible that $\beta'(\alpha) = 0$ at some point, and conseguentely we find $\beta(\alpha_*) = 0$, where $\alpha^* > 0$, and we can see the plot in figure 7.9a, the IR fixed point, and in figure 7.9b, the UV fixed point.

When α approaches this value, α_*, the renormalization group flow slows to a halt; thus $\alpha = \alpha_*$ would be a nontrivial fixed point of the renormalization group. In this model, the running coupling constant α tends to α_* in the limit of large momentum.

The non-zero value of $\beta(\alpha)$ is what breaks the conformal symmetry, introducing a physical scale into the coupling constant, so one starts from a conformal theory at the origin. It can be seen that, by following the trajectories of $\beta(\alpha)$, one can reach a point where it vanishes again, i.e., a new conformal theory, usually different from the previous one, given that the coupling constant has changed its value (there might be bound states that better describe the new theory).

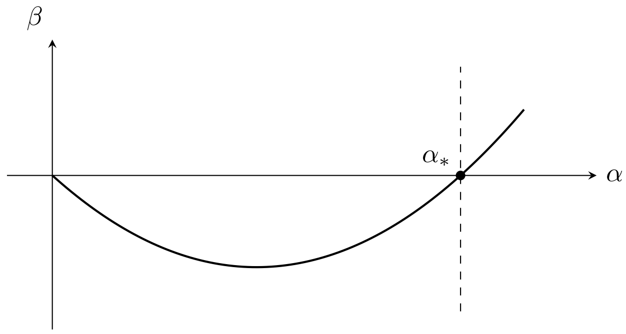
At the fixed point α^* the β in the equation (7.5.25):

$$\int_{\alpha(\mu_0)}^{\alpha(\mu)} \frac{\mathrm{d}\alpha'}{\beta(\alpha')} = \int_{\mu_0}^{\mu} \frac{\mathrm{d}\mu'}{\mu'} = \log\left(\frac{\mu}{\mu_0}\right) \tag{7.6.26}$$

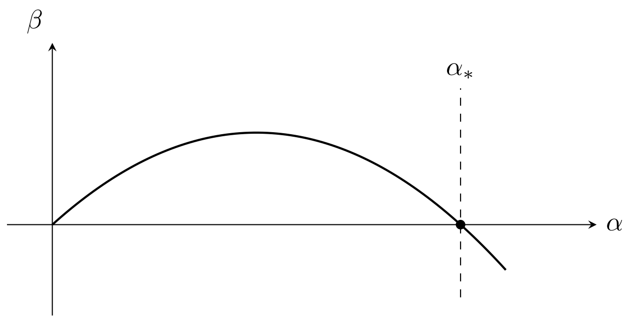
vanishes, so there is a positive pole; the only way for the value of μ to repeat the same behavior is to have $\mu \to \infty$, which is why it is called an ultraviolet fixed point (figure 7.9b).

In case in figure 7.9a, the value of beta is negative, so the only way to reproduce a negative pole is $\mu \to 0$, hence an infrared fixed point.

They are called fixed points because the renormalization group flow causes the coupling constant to increase (figure 7.9b) or decrease (figure 7.9a) until it reaches α^*, where the beta function vanishes and the coupling constant no longer changes, meaning the value of beta remains fixed.



(a) IR fixed point.



(b) UV fixed point. It's useful for studing phase transitions in e.g. Ising model.

Figure 7.9

If we assume such a point does exsist we can infer the properties of the theory close to the fixed point. Taking the case in figure 7.9b the β behaves in the vicinity of the fixed point as:

$$\beta(\alpha) = -B(\alpha - \alpha_*) \tag{7.6.27}$$

where B is a constant. We can see, from the equation (7.5.62):

$$\int_{\alpha(\mu)}^{\alpha(|p|)} \frac{\mathrm{d}\alpha}{\alpha - \alpha_*} = -B \log\left(\frac{|p|}{\mu}\right) \tag{7.6.28}$$

for same scales $|p|$ and μ. We get the solution:

$$\implies \quad \log\left(\frac{\alpha(|p|)-\alpha_*}{\alpha(\mu)-\alpha_*}\right) = \log\left(\left(\frac{|p|}{\mu}\right)^{-B}\right) \tag{7.6.29}$$

$$\implies \quad \alpha(|p|) = \alpha_* + (\alpha(\mu)-\alpha_*)\left(\frac{|p|}{\mu}\right)^{-B} \tag{7.6.30}$$

so we find an exact solution to the evolution of the coupling. Thus, α indeed tends to α_* as $p \to \infty$, and the rate of approach is governed by the slope of the β function at the fixed point.

We can also look at the solution to the 2-point function from the CS equation. Indeed, this behavior has a dramatic consequence for the exact solution (7.5.61) of the Callan-Symanzik equation for $\tilde{G}(p)$. For p sufficiently large, the integral in the exponential factor in this equation will be dominated by values of p for which $\alpha(p)$ is close to α_*. Taking:

$$\gamma_* = \gamma(\alpha(|p|;\alpha(\mu))) \tag{7.6.31}$$

we have:

$$\tilde{G}_2(p^2,\mu^2,\alpha_*) = \frac{-i}{p^2} g_2'(\alpha_*) \exp\left\{2\int_\mu^{|p|} \frac{\mathrm{d}|p'|}{|p'|}\gamma_*\right\} \tag{7.6.32}$$

$$= \frac{-i}{p^2}\left(\frac{\mu^2}{p^2}\right)^{-\gamma_*} g_2'(\alpha_*). \tag{7.6.33}$$

Thus the two-point correlation function returns to the form of a simple scaling law, but with a power law different from that expected by dimensional analysis. At the fixed point we have a scale-invariant quantum field theory in which the interactions of the theory affect the law of rescaling. We've saw again a motivation for naming *anomalous dimension*, like the equation (7.5.14).

A similar behavior is possible in an asymptotically free theory. If the β function has the form shown in figure 7.9a, the running coupling constant will tend to a fixed point α_* as $p \to 0$. The two-point correlation function of fields $\tilde{G}(p)$ will tend to a power law as in (7.6.33) for asymptotically small momenta.

The two cases shown in figures 7.9a and 7.9b are called, respectively, ultraviolet-stable and infrared-stable fixed points.

7.7 Evolution of mass parameters

Finally, we discuss the renormalization group for theories with masses. We note, though, that although we treat these masses as arbitrary parameters, we will continue to use renormalization conventions that are independent of mass, and we will often treat the masses as small parameters. This approach breaks down at momentum scales much less than the scale of masses, but it is sufficient, and simpler than alternative approaches, for most practical applications of the renormalization group.

Let's consider a massive scalar theory and look at the CS equation (7.3.16) for an on-shell ($p_i^2 = m^2$) S-matrix element. In the on-shell case the factor of $n\gamma_\phi(\lambda)$ does not contribute (LSZ) and we have:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + \gamma_m(\lambda)m\frac{\partial}{\partial m}\right) S = 0 \tag{7.7.1}$$

where:

$$S \equiv S(p_i \cdot p_j, m; \mu, \lambda(\mu)) \tag{7.7.2}$$

with $p_i \cdot p_j$ the momentum invariants. We can normalize S to form a dimensionless function $\hat{S}$ using the scale μ:

$$S = \mu^{-d}\hat{S}\left(\frac{p_i \cdot p_j}{\mu^2}, \frac{m^2}{\mu^2}, \lambda(\mu)\right). \tag{7.7.3}$$

We can perform a simple dimensional analysis of S, taking a single invariant Q^2 for simplicity:

$$\hat{S} \equiv \hat{S}\left(\frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}, \lambda(\mu)\right). \tag{7.7.4}$$

So we get the equation:

$$\left(\mu\frac{\partial}{\partial\mu} + Q\frac{\partial}{\partial Q} + m\frac{\partial}{\partial m} + d\right) S = \tag{7.7.5}$$

$$= 2\left(\mu^2\frac{\partial}{\partial\mu^2} + Q^2\frac{\partial}{\partial Q^2} + m^2\frac{\partial}{\partial m^2} + \frac{d}{2}\right)(\mu^2)^{-d/2}\hat{S} \tag{7.7.6}$$

$$= 2(\mu^2)^{-d/2}\left(\mu^2\frac{\partial}{\partial\mu^2} + Q^2\frac{\partial}{\partial Q^2} + m^2\frac{\partial}{\partial m^2}\right)\hat{S} \tag{7.7.7}$$

$$= 0. \tag{7.7.8}$$

Combining (7.7.5) minus (7.7.1) gives:

$$\left(Q\frac{\partial}{\partial Q} - \beta(\lambda)\frac{\partial}{\partial\lambda} + (1-\gamma_m(\lambda))m\frac{\partial}{\partial m} + d\right) S = 0. \tag{7.7.9}$$

This equation may be solved using the same methods has used in section §7.5 and leads to a solution in which both coupling and mass run with energy:

$$\hat{\lambda}(Q;\lambda) \qquad \text{satisfying} \qquad Q\frac{\mathrm{d}\hat{\lambda}}{\mathrm{d}Q} = \beta(\hat{\lambda}) \tag{7.7.10}$$

$$\hat{m}(Q;\lambda) \qquad \text{satisfying} \qquad Q\frac{\mathrm{d}\hat{m}}{\mathrm{d}Q} = -(1-\gamma_m(\hat{\lambda}))\hat{m} \tag{7.7.11}$$

where $\hat{\lambda}$ runs from the point λ where we have measured the coupling and fixed the renormalization scheme. The mass runs from the value of the mass at the same renormalization scale, μ for example:

$$\hat{\lambda}(\mu; \lambda) = \lambda(\mu) \tag{7.7.12}$$

$$\hat{m}(\mu; \lambda) = m(\mu) \tag{7.7.13}$$

then:

$$\hat{m}(Q; \lambda) = m(\mu) \exp\left\{ - \int_{\mu}^{Q} \frac{\mathrm{d}Q'}{Q'} (1 - \gamma_m(\hat{\lambda})) \right\}. \tag{7.7.14}$$

Chapter 8

Unitarity and gauge invariance

Unitarity of the S-matrix leads to powerful constraints on the discontinuities of loop amplitudes obtained from lower order amplitudes. We will compute discontinuities/generalised discontinuities via Cutkovsky rules, and that will leads to *generalized unitarity* constraints, that can be used to determine the coefficients of basis loop integrals for any amplitude.

The unitarity of the S-matrix for physical states relies on the cancellation of unphysical degrees of freedom (ghosts and longitudinal polarisations). This can be demostrated using BRST symmetry.

8.1 $SS^\dagger = 1$ and implications

In the Heisemberg picture, where operators are time dependent and states are time independent, we may take (postulate) two orthonormal bases for the Hilbert (or Fock) space:

$$|\alpha, \text{in}\rangle \qquad \text{observations at } t \to -\infty \tag{8.1.1}$$

$$|\beta, \text{out}\rangle \qquad \text{observations at } t \to +\infty. \tag{8.1.2}$$

The scattering matrix S elements are defined as:

$$S_{\beta\alpha} = \langle \beta, \text{out} | \alpha, \text{in}\rangle \tag{8.1.3}$$

and, since *in* states are orthonormal, we have:

$$\langle \alpha, \text{in} | \beta, \text{in}\rangle = \delta_{\alpha\beta}. \tag{8.1.4}$$

We may insert a complete set of *out* states in the relation above[1], to find:

$$\sum_k \langle \alpha, \mathrm{in}|k, \mathrm{out}\rangle \langle k, \mathrm{out}|\beta, \mathrm{in}\rangle = \sum_k (S_{k\alpha})^\dagger S_{k\beta} \tag{8.1.6}$$

$$= \delta_{\alpha\beta} \tag{8.1.7}$$

or simply:

$$S^\dagger S = SS^\dagger = 1. \tag{8.1.8}$$

So, we found that the S matrix is *unitary*. Unitarity of the S matrix reflects the fundamental principle of probability conservation. Even though we have to introduce in certain instances the artificial device of an indefinite metric in Hilbert space, the physical quantities always refer to states with positive norm, preserved through the time evolution. If the relation (8.1.8) is not satisfied the probability is not conserved or the theory left out some states that are physical.

Now, we can see the *optical theorem*, which connects this relation to the fact that the immaginary part of the forward scattering amplitude is related to the cross-section. Indeed, when we separate the scattering process in two parts (8.1.9), the identity $\mathbb{1}$, so the situation where the initial and final states are the same, and the second part iT, where the collision could happen, means that every time something dissapear during scattering process (from the initial state), it's is related to the immaginary part; but, if the theory is unitary, what is dissapearing has to be related to the final state. We do this because we want to see that there is a more general connection with the *discontinuities* of loop amplitudes. Let:

$$S = \mathbb{1} + iT \tag{8.1.9}$$

where T is the transition matrix. Inserting this into (8.1.8) leads to:

$$-i(T - T^\dagger) = TT^\dagger = T^\dagger T. \tag{8.1.10}$$

The relation (8.1.10) connects different perturbative orders. The scattering amplitude $\mathcal{A}$ from a state I to a state F is defined by:

$$\langle F|\, T\, |I\rangle = i(2\pi)^4 \delta^{(4)}(p_I - p_F)\mathcal{A}(I \to F) \tag{8.1.11}$$

[1]Note that the correct relation of completation is:

$$\sum\!\!\!\!\!\!\!\int_k |k, \mathrm{out}\rangle \langle k, \mathrm{out}| = \mathbb{1} \tag{8.1.5}$$

so with a sum over all the continuous variables and not only with the sum over the discrete quantum numbers.

where p_I is the total incoming momentum and p_F is the total outgoing. Combining (8.1.10) and (8.1.11) gives:

$$\begin{aligned} -i \langle F| T |I\rangle + i \langle F| T^\dagger |I\rangle &= \langle F| T^\dagger T |I\rangle && (8.1.12)\\ &= \sum\!\!\!\!\!\!\!\!\int_k \langle F| T^\dagger |k\rangle \langle k| T |I\rangle && (8.1.13)\end{aligned}$$

where we use:

$$\sum\!\!\!\!\!\!\!\!\int_k = \sum_{n=2}^{\infty} \int \prod_{j=1}^{n} \frac{\mathrm{d}^3\vec{k}_j}{(2\pi)^3 2E_j} \quad , \quad E_j^2 = |\vec{k}_j|^2 + m_j^2 \tag{8.1.14}$$

and we get:

$$(2\pi)^4\delta^{(4)}(p_I - p_F)\mathcal{A}(I \to F) - (2\pi)^4\delta^{(4)}(p_I - p_F)\mathcal{A}(F \to I) \tag{8.1.15}$$

$$= \sum\!\!\!\!\!\!\!\!\int_k \Big(i(2\pi)^4\delta^{(4)}(p_k - p_F)\mathcal{A}(F \to k)\Big)^\dagger (2\pi)^4\delta^{(4)}(p_k - p_I)\mathcal{A}(I \to k) \tag{8.1.16}$$

$$= \sum\!\!\!\!\!\!\!\!\int_k (2\pi)^8\delta^{(4)}(p_k - p_F)\delta^{(4)}(p_k - p_I)\mathcal{A}(F \to k)^\dagger\mathcal{A}(I \to k) \tag{8.1.17}$$

$$= \sum\!\!\!\!\!\!\!\!\int_k (2\pi)^8\delta^{(4)}(p_I - p_F)\delta^{(4)}(p_k - p_I)\mathcal{A}(F \to k)^\dagger\mathcal{A}(I \to k) \tag{8.1.18}$$

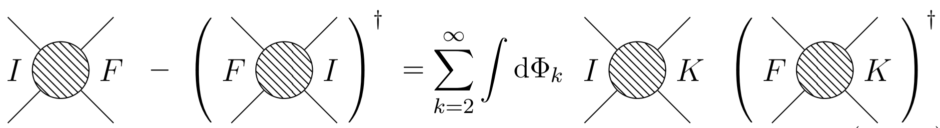
hence:

$$\mathcal{A}(I \to F) - \mathcal{A}(F \to I)^\dagger = \sum\!\!\!\!\!\!\!\!\int_k (2\pi)^4\delta^{(4)}(p_k - p_I)\mathcal{A}(F \to k)^\dagger\mathcal{A}(I \to k). \tag{8.1.19}$$

If we set $F = I$ we recover the optical theorem:

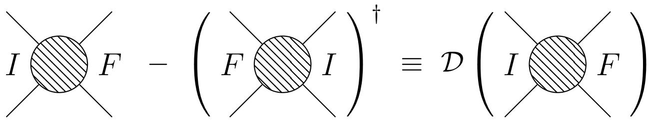
$$\begin{aligned} 2i\,\mathrm{Im}\{\mathcal{A}(I \to I)\} &= \sum\!\!\!\!\!\!\!\!\int_k (2\pi)^4\delta^{(4)}(p_k - p_I)|\mathcal{A}(I \to k)|^2 && (8.1.20)\\ &= \sum_{k=2}^{\infty}\int \mathrm{d}\Phi_k(p_I; p_1, \ldots, p_k)\,|\mathcal{A}(I \to k)|^2 && (8.1.21)\end{aligned}$$

where we wrote explicitly the k-particle phase-space integral. We may represent relation (8.1.19) graphically by:



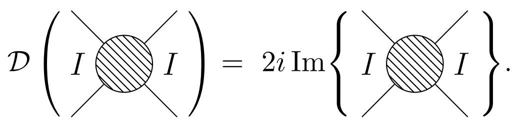
(8.1.22)

where we can define the operation:



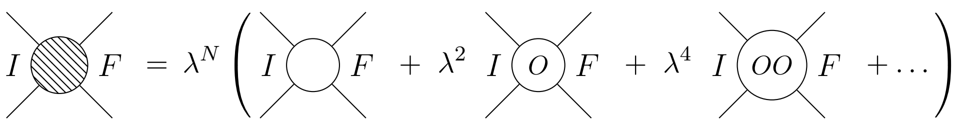
(8.1.23)

e.g.:

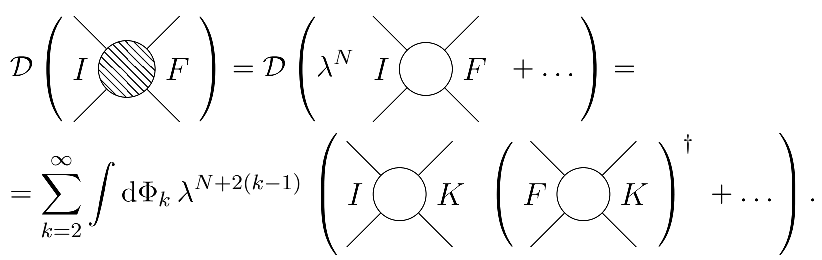


(8.1.24)

The amplitudes can be expanded perturbatively. If we assume a theory with a 3-point coupling λ, then the leading order will be λ^n where $N = \#I + \#F - 2$. Therefore:

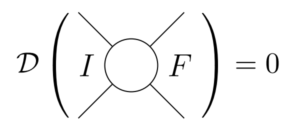


that implies the relation (8.1.22):

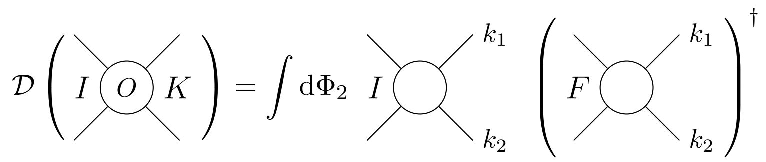


(8.1.25)

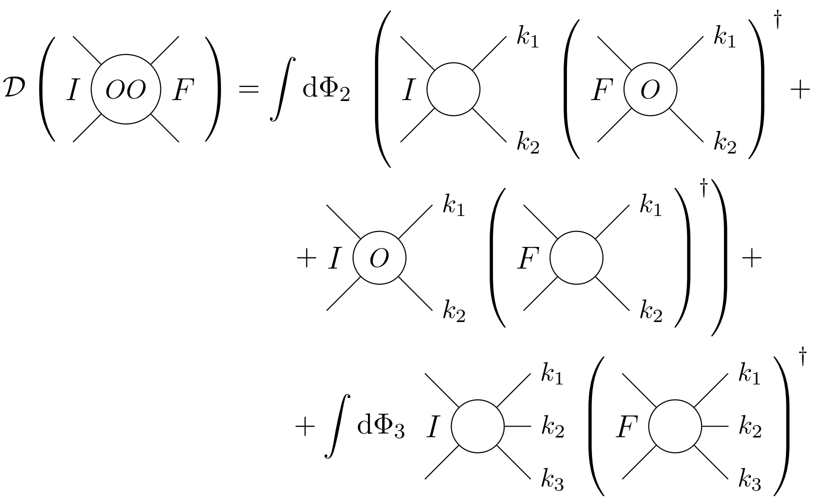
Comparing order by order in λ gives:



(8.1.26)



(8.1.27)



(8.1.28)

so we find relations between amplitudes at different loop orders. The operation $\mathcal{D}$ is not $2i\,\mathrm{Im}$ in general, since the amplitude contains *branch cuts* which lead to *discontinuities.* Let's look at a particular example to understand better.

8.1.1 Discontinuities of Feynman diagrams

First of all, what does it mean that a Feynman diagram has discontinuities?

Referring to chapter §21.2 from Nastase Let no one ignorant of geometry enter my door.Nastase, we can note that above the reduced-amplitude $\mathcal{M}$ is defined formally, but we can define it purely by Feynman diagrams, and that is the definition we will use below, since we want to prove unitarity for Feynman diagrams. Then, $\mathcal{M}$ is defined at some center of mass energy, with $s = E_{cm}^2 \in \mathbb{R}^+$. But we analytically continue to an analytic function $\mathcal{M}(s)$ defined on the complex s plane.

Consider $\sqrt{s_0}$ the threshold energy for production of the lightest multiparticle state. Then, if $s < s_0$ and real, it means that any possible intermediate state created from the initial state cannot go on-shell, and can only be *virtual.* In turn, this means that any propagators in the expression of $\mathcal{M}$ in terms of Feynman diagrams are always nonzero, leading to an $\mathcal{M}(s)$ real for

$s < s_0$ real, that is:

$$\mathcal{M}(s) = [\mathcal{M}(s^*)]^*. \tag{8.1.29}$$

We then analytically continue to $s > s_0$ real, and since both sides of the above relation are analytical functions, we apply it for $s \pm i\epsilon$ with $s > s_0$ real, giving (since now the propagators can be on-shell, so we need to avoid the poles by moving s a bit away from the real axis):

$$\mathrm{Re}\{\mathcal{M}(s + i\epsilon)\} = \mathrm{Re}\{\mathcal{M}(s - i\epsilon)\} \tag{8.1.30}$$

$$\mathrm{Im}\{\mathcal{M}(s + i\epsilon)\} = -\,\mathrm{Im}\{\mathcal{M}(s - i\epsilon)\}. \tag{8.1.31}$$

This means that for $s > s_0$ and real there is a branch cut in s (by moving accross it, we go between two Riemann sheets), with discontinuity:

$$\mathrm{Disc}_s\mathcal{M} = 2i\,\mathrm{Im}\{\mathcal{M}(s + i\epsilon)\}. \tag{8.1.32}$$

It is much easier to compute the discontinuity only rather than the full amplitude for complex s, taking the discontinuity.

The discontinuity of Feynman diagrams with loops, across cuts on loops (equal from the above to the imaginary parts of the same diagrams) will be related to $\int d\Pi|\mathcal{M}|^2$, that is the optical theorem formula. We will obtain the so-called *cutting rules*, ironically found by Cutkovsky, so also called the Cutkovsky rules.

Let's see a pratical example before the physics. The discontinuity of an amplitude $\mathcal{A}$ across the p^2-channel is defined by:

$$\lim_{\epsilon\to 0}\Big(\mathcal{A}(\dots, p^2 + i\epsilon, \dots) - \mathcal{A}(\dots, p^2 - i\epsilon, \dots)\Big) \equiv \mathrm{Disc}_{p^2}(\mathcal{A}). \tag{8.1.33}$$

It is useful to look at the branch cut in the logarithm to see how this is connected to the operation $\mathcal{D}$:

$$\mathrm{Disc}_x(\log(x)) = \lim_{\epsilon\to 0}\Big(\log(x + i\epsilon) - \log(x - i\epsilon)\Big) \tag{8.1.34}$$

$$= \begin{cases} 0 & , \quad x > 0 \\ 2\pi i & , \quad x < 0. \end{cases} \tag{8.1.35}$$

For $x < 0$ we have $\log(x) = \log(|x|) + i\pi$ so we may write:

$$2i\,\mathrm{Im}\{\log(x)\} = \mathrm{Disc}_x(\log(x)) \qquad , \quad x < 0. \tag{8.1.36}$$

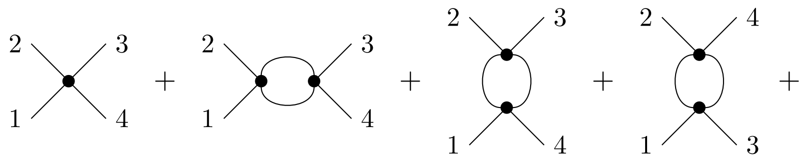
This helps to justify that for general states I, F the operation $\mathcal{D}$ *is the discontinuity of the amplitude* in the $I \to F$ channel.

For concreteness let's choose a $2 \to 2$ scattering amplitude $\mathcal{A}(p_1p_2 \to p_3p_4)$ and consider the discontinuity in the $s_{12} = (p_1+p_2)^2$ channel:

$$\mathrm{Disc}_{s_{12}}\Big(\mathcal{A}(p_1p_2 \to p_3p_4)\Big) = \mathcal{A}(p_1p_2 \to p_3p_4) - \mathcal{A}(p_3p_4 \to p_1p_2)^\dagger \tag{8.1.37}$$

$$= \sum_{k=1}^{\infty} \int \mathrm{d}\Phi_k \; \mathcal{A}(p_1p_2 \to \{k\})\big(\mathcal{A}(p_3p_4 \to \{k\})\big)^\dagger. \tag{8.1.38}$$

In ϕ^4 theory we can even explicitly write the amplitude up to 1-loops as:

$\mathcal{A}_4 =$ 

$$+ \text{ counter-terms } + \mathcal{O}(\lambda^4) \tag{8.1.39}$$

so:

$$i\mathcal{A}_4^{[d]} = \lambda + \frac{\lambda^2}{2}\left[\hat{I}_2^{[d]}\left(\frac{s_{12}}{m^2}, \frac{\mu_R^2}{m^2}\right) + \hat{I}_2^{[d]}\left(\frac{s_{14}}{m^2}, \frac{\mu_R^2}{m^2}\right) + \hat{I}_2^{[d]}\left(\frac{s_{13}}{m^2}, \frac{\mu_R^2}{m^2}\right)\right] + \mathcal{O}(\lambda^4) \tag{8.1.40}$$

where we write:

$$\hat{I}_2^{[d]} = -i\mu_R^{2\epsilon} I_2\Big|_{\text{finite}} \tag{8.1.41}$$

in a minimal scheme. We already know, using the result (3.2.56) and the relative counterterm:

$$\hat{I}_2^{[d]}\left(\frac{s_{12}}{m^2}, \frac{\mu_R^2}{m^2}\right) = \frac{1}{(4\pi)^2} C_\Gamma \left(\frac{\mu_R^2}{m^2}\right)^\epsilon \left(\frac{1}{\epsilon} + 2 - \beta \log\left(\frac{-\beta_+}{\beta_-}\right)\right) - \frac{1}{(4\pi)^2}\frac{C_\Gamma}{\epsilon} \tag{8.1.42}$$

$$= \frac{1}{(4\pi)^2}\left(2 - \beta \log\left(\frac{-\beta_+}{\beta_-}\right) + \log\left(\frac{\mu_R^2}{m^2}\right)\right) \tag{8.1.43}$$

with:

$$\beta = \sqrt{1 - \frac{4m^2}{s_{12}}}. \tag{8.1.44}$$

The relevant discontinuity in $\mathcal{A}_4$ in the s_{12}-channels comes from the s_{12} bubble integral so we can explicitly study the discontinuity via poles in the integrand. We have:

$$I_2^{[d]}(p^2, m^2) = \int_k \frac{1}{((k-p/2)^2 - m^2 + i\emptyset^+)((k+p/2)^2 - m^2 + i\emptyset^+)} \tag{8.1.45}$$

which contains four possible poles. See the figure 8.1. We can already see that this amplitude has a discontinuity for $k_0 > 2m$ (in the physical region, since with center of mass energy k_0 we can create an on-shell two-particle intermediate state of energy $2m$ only if $k_0 > 2m$), but we compute it for $k_0 < 2m$ and then analytically continue. Explicitly, choosing the frame $p = (p^0, \vec{\emptyset})$, we have:

$$(k \pm p/2)^2 - m^2 + i\emptyset^+ = (k^0)^2 - |\vec{k}|^2 \pm k^0 p^0 + \frac{(p^0)^2}{4} - m^2 + i\emptyset^+ \tag{8.1.46}$$

$$= (k^0)^2 \pm k^0 p^0 + B \tag{8.1.47}$$

$$= (k_0 - k_\pm^+)(k_0 - k_\pm^-) \tag{8.1.48}$$

where we use:

$$B = -|\vec{k}|^2 + \frac{(p^0)^2}{4} - m^2 + i\emptyset \tag{8.1.49}$$

$$k_\delta^\sigma = \frac{1}{2}\left(\delta p^0 + \sigma\sqrt{p_0^2 - 4B}\right). \tag{8.1.50}$$

The poles in the complex k^0 plane are located, as already said, in the figure 8.1.

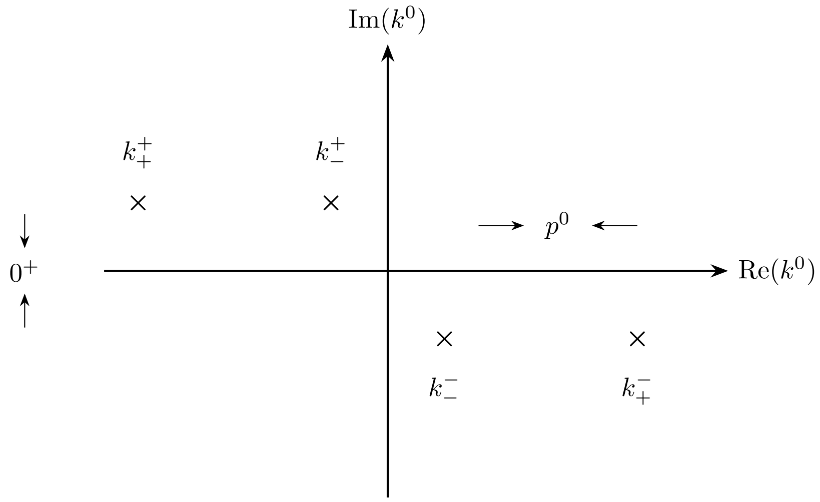


Figure 8.1

We now perform the k^0 integration ($4d$ only):

$$\int_k \longrightarrow \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \oint_\gamma \frac{\mathrm{d}k^0}{2\pi} \tag{8.1.51}$$

using the residue theorem. You can see che curve and the regions in figure

8.2. At $k^0 = k_-^-$ we have:

$$\mathrm{Res}_{k^0=k_-^-}(I_2^{[4]}) = \frac{1}{2\pi}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\frac{1}{(k_-^- - k_+^-)(k_-^- - k_+^+)(k_-^- - k_-^+)} \tag{8.1.52}$$

$$= \frac{1}{2\pi}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\frac{1}{(-2E_k + i\emptyset^+)p_0(-2E_k + p_0 + i\emptyset^+)} \tag{8.1.53}$$

where $E_k = |\vec{k}|^2 + m^2$. Since:

$$\frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} = \int \frac{\mathrm{d}|\vec{k}|\ |\vec{k}|^2\ \mathrm{d}^2\Omega}{(2\pi)^3} \tag{8.1.54}$$

$$= \int_m^\infty \frac{\mathrm{d}E_k\ |\vec{k}|\ E_k\ 4\pi}{(2\pi)^3} \tag{8.1.55}$$

we can conclude:

$$\mathrm{Res}_{k^0=k_-^-}(I_2^{[4]}) = \frac{4\pi}{(2\pi)^4}\int_m^\infty \frac{\mathrm{d}E_k\ E_k\,|\vec{k}|}{p_0(2E_k - i\emptyset^+)(2E_k - p_0 - i\emptyset^+)}. \tag{8.1.56}$$

The residue has a brach cut *inside* the integration region if $p_0 > 2m$. Indeed, we can notice that if $k_0 < 2m$, the denominator in the last line has no pole on the integral path, as $k_0 < 2m < 2E_k$, so the residue is real (has no discontinuity). But if $p_0 > 2m$, we have a pole on the path of integration, so there is a discontinuity between $k^2 + i\emptyset^+$ and $k^2 - i\epsilon$. To find it, we can compute the Disc directly using the Cauchy principle value, P:

$$\frac{1}{2E_k - p_0 \pm i\eta} = P\left(\frac{1}{p_0 - 2E_k}\right) \mp i\pi\delta(2E_k - p_0) \tag{8.1.57}$$

in this way we get:

$$\mathrm{Disc}_{p^2}\Big(\mathrm{Res}_{k^0=k_-^-}(I_2^{[4]})\Big) = \lim_{\eta\to 0}\left(\mathrm{Res}_{k^0=k_-^-}(I_2^{[4]})\bigg|_{+i\eta} - \mathrm{Res}_{k^0=k_-^-}(I_2^{[4]})\bigg|_{-i\eta}\right) \tag{8.1.58}$$

$$= \frac{2}{(2\pi)^3}\int_m^\infty \frac{\mathrm{d}E_k\ E_k\,|\vec{k}|}{p_0 2E_k}(-2\pi i\delta(p_0 - 2E_k)) \tag{8.1.59}$$

$$= i\frac{\beta}{(4\pi)^2}. \tag{8.1.60}$$

NB. The discontinuity can be computed directly using the replacement:

$$\frac{1}{p_0 - 2E_k + i\emptyset^+} \longrightarrow 2\pi i\delta(p_0 - 2E_k). \tag{8.1.61}$$

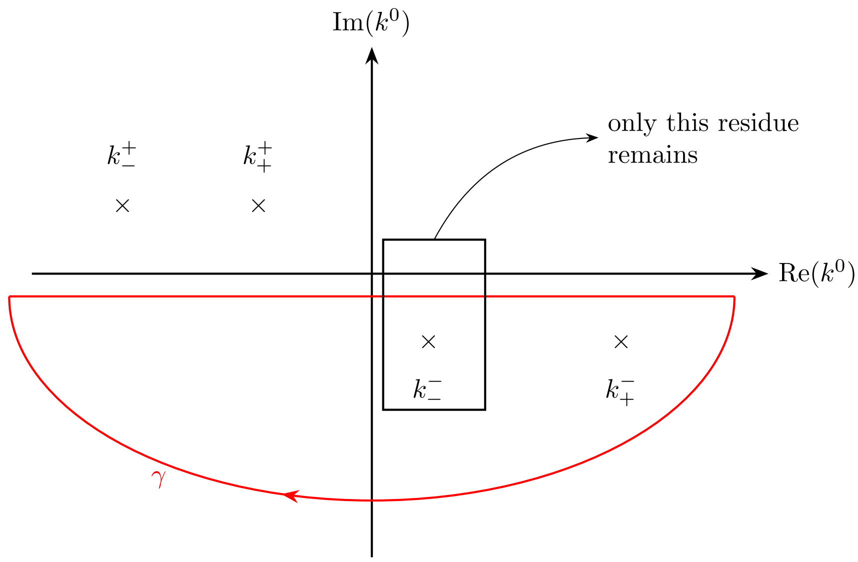


Figure 8.2

For the discontinuity of the full integral we need to sum over all residues inside a closed contour, however it turns out, after closing the contour at $-i\infty$, that the residue at k_+^- does not have a branch cut in the integration region (outside $p^0 > 2m$). See figure 8.2.

So, to conclude we can write:

$$\begin{aligned}\mathrm{Disc}_{p^2}\left(I_2^{[4]}(p^2,m^2)\right) &= 2\pi i\ \mathrm{Disc}_{p^2}\left(\mathrm{Res}_{k^0=k_-^-}(I_2^{[4]}(p^2,m^2))\right) && (8.1.62)\\ &= 2\pi i\left(\frac{i\beta}{16\pi^2}\right) && (8.1.63)\\ &= -\frac{\beta}{8\pi}. && (8.1.64)\end{aligned}$$

Observations. We arrive at the same result by applying the rule:

$$\begin{aligned}\frac{1}{(k\pm p/2)^2 - m^2 + i\emptyset^+} &\longrightarrow -2\pi i\,\delta((k\pm p/2)^2 - m^2)\,\theta(\pm k^0 + p/2) && (8.1.65)\\ &= -2\pi i\,\delta^{(+)}((k\pm p/2)^2 - m^2). && (8.1.66)\end{aligned}$$

What we've already saw, the fact that the propagator (which contains poles) can be substituted with on-shell delta function, is the *Cutkovsky rule* and can be used to prove the optical thoerem.

> **Exercise 17.** Check, using the closed form for $I_2^{[4-2\epsilon]}(p^2,m^2)$, that the branch cut in $\log(-\beta_+/\beta_-)$ mathches the result obtained above.

We can now continue the analysis of the amplitude since:

$$\mathrm{Disc}_{s_{12}}\Big(\mathcal{A}(p_1p_2 \to p_3p_4)\Big) = -i\frac{\lambda^2}{2}\,\mathrm{Disc}_{s_{12}}\left(\hat{I}_2\left(\frac{s_{12}}{m^2},\frac{\mu_R^2}{m^2}\right)\right) \tag{8.1.67}$$
$$= -\frac{\lambda^2}{2}\left(-\frac{\beta}{8\pi}\right) \tag{8.1.68}$$
$$= \frac{\lambda^2\beta}{16\pi}. \tag{8.1.69}$$

Since the initial and final states are the same ($\phi\phi \to \phi\phi$), the optical theorem applies.

Exercise 18. Confirm using the final state averaged tree-level amplitude squared:

$$\langle|\mathcal{A}_4^{(0)}|^2\rangle = \frac{1}{2}\lambda^2 \tag{8.1.70}$$
$$\int \mathrm{d}\Phi_2\,\langle|\mathcal{A}_4^{(0)}|^2\rangle = \frac{\lambda^2\beta}{16\pi}. \tag{8.1.71}$$

8.2 Generalized unitarity and the Cutkovsky rules

We saw that the discontinuity of a Feynman integral could be computed using the Cutkovsky rules where propagators are replaced with on-shell delta functions. In general we can give the *general cutting rules* (in general valid at all loops):

1. Cut the diagram in all possible ways such that cut propagators can be put simultaneously on-shell.

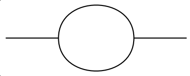
2. For each cut, replace:

$$\frac{1}{k^2 - m^2 + i\emptyset^+} \longrightarrow -2\pi i\,\delta^{(+)}(k^2 - m^2) \tag{8.2.1}$$

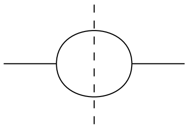
   in the propagator, and then do the integral.

3. Sum over all possible cuts.

Notice we can cut a diagram only if every external propagator can be put on-shell without breaks the momentum conservation rule. Doing (8.2.1) we can write:

$$\mathrm{Disc}_{p^2}(I_2(p^2,m^2)) = \mathrm{Disc}_{p^2}\left(\right. \tag{8.2.2}$$


$$\left. \right)$$

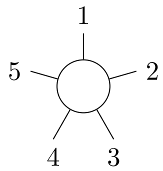
$$= \tag{8.2.3}$$


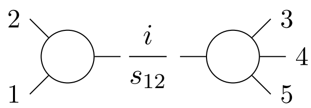
$$= -\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,(2\pi)^2\,\delta(k^2-m^2)\,\delta((k-p)^2-m^2) \tag{8.2.4}$$

$$= -\frac{\beta}{8\pi} \tag{8.2.5}$$

where we call (8.2.3) the *2-particle cut integral.* While we lose connection with unitarity, we can replace any propagator in an amplitude with a on-shell δ-function (obviously with the correct momentum in it) and obtain a *generalised discontinuity* of the amplitude.

Amplitudes *factorize* when intermediate propagators go on-shell, e.g.:

$$\lim_{s_{12}\to 0}\left(\right. \tag{8.2.6}$$


$$\left. \right) = \quad \frac{i}{s_{12}}$$


where the diagrams on the right side are on-shell. This is trivial in scalar theories, but in gauge relies on the spin sum relations:

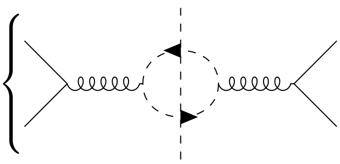
$$\not{p} + m \ \overset{p^2\to m^2}{\longrightarrow}\ \sum_s u_s\overline{u}_s \tag{8.2.7}$$

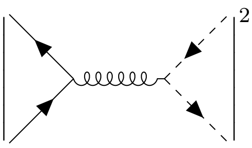
$$-g^{\mu\nu} + \frac{p^\mu n^\nu + n^\mu p^\nu}{p\cdot n} \ \overset{p^2\to m^2}{\longrightarrow}\ \sum_n \epsilon_n\epsilon_{-n} \tag{8.2.8}$$

where the last relation is written with the *light-like axial gauge.* Hence, when apply cuts to internal lines we may facturize the integrand of any amplitude into products of lower order on-shell amplitudes.

The application of multiple cuts to constrain the form of a scattering amplitude is commonly referred to as *generalized unitarity.*

Using the Cutkovsky rules it proved unitarity (equivalent to the optical theorem) in perturbation theory. It's convenient use the *cuts* (as we did in MSc course *Phenomenology of Fundamental Interaction*) because we can calculate "easier" something like:

$$2\,\mathrm{Im}\left\{ \right. \tag{8.2.9}$$


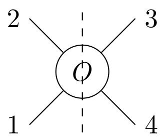
$$\left. \right\} = \not{\sum}\ \Bigg| \qquad \Bigg|^2 .$$


There exists a more general method, the method of the *largest time equation* (which is a generalization of the above method), which was used by 't Hooft and Veltman to prove the perturbative unitarity of spontaneously broken gauge theories.

8.2.1 Double-cuts and dispersion relations

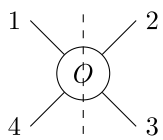
Let's try to use double cut information to construct the 1-loop $\phi\phi \to \phi\phi$ amplitude. We already saw, with (8.1.69), that:

$$\mathrm{Disc}_{s_{12}}(\mathcal{A}_4^{(1)}) = \frac{\lambda^2}{16\pi}\sqrt{1-\frac{4m^2}{s_{12}}} \tag{8.2.10}$$

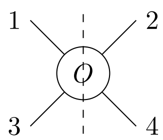
$$=$$ 

(8.2.11)

The cuts in the other channels, s_{14} and s_{13}, are symmetric:



$$= \frac{\lambda^2}{16\pi}\sqrt{1-\frac{4m^2}{s_{14}}} \tag{8.2.12}$$



$$= \frac{\lambda^2}{16\pi}\sqrt{1-\frac{4m^2}{s_{13}}}. \tag{8.2.13}$$

To obtain information at the amplitude level we could try to perform dispersion integrals:

$$f(s) = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}s'}{s-s'}\mathrm{Disc}_{s'} f(s'). \tag{8.2.14}$$

Since we have a multivariate function this route in a bit difficult but can be done *summing over all cut contributions*. There is however a simpler ruoute if we identify a *basis* of integrals for the amplitude. The scalar amplitude is trivial, there's no tensor reduction necessary and we can identify from the diagrams by eye that:

$$\mathcal{A}_4^{(1)} = c_{12} I_2(s_{12}, m^2) + c_{23} I_2(s_{23}, m^2) + c_{13} I_2(s_{13}, m^2). \tag{8.2.15}$$

From which we use the *cut* operation to project out the *rational coefficients*, e.g.:

$$\mathrm{Cut}_{12}(\mathcal{A}_4^{(1)}) = c_{12}\mathrm{Cut}_{12}(I_2(s_{12,m^2})) \tag{8.2.16}$$

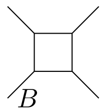
$$\frac{\lambda^2\beta}{16\pi} = c_{12}\left(-\frac{\beta}{8\pi}\right) \tag{8.2.17}$$

$$\implies \quad c_{12} = -\frac{\lambda^2}{2}. \tag{8.2.18}$$

8.2.2 General one-loop amplitudes

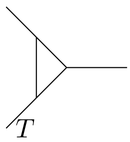
It is possible to show that any one-loop amplitude can, via tensor reduction, be expressed as a linear combination of scalar integrals (usually called *master integrals*). If we take external particles to live in $4d$ then the maximum numbers of independent propagators is four, therefore (massless case):

$$\mathcal{A}^{(1)}(p_1, \ldots, p_n) = \sum_{B \in \text{Boxed}} C_B$$



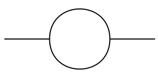
$$[1] +$$

$$+ \sum_{T \in \text{Triangles}} C_T$$



$$[1] +$$

$$+ \sum_{b \in \text{Bubbles}} C_b$$



$$[1] +$$

$$+ R + \mathcal{O}(\epsilon) \tag{8.2.19}$$

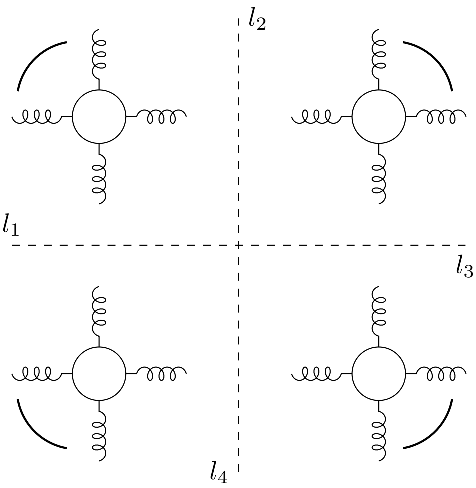
R is a potential rational coefficient coming from:

$$C \cdot I = (C_0 + \epsilon C_1 + \ldots)\left(\frac{I_{-1}}{\epsilon} + I_0 + \ldots\right) \tag{8.2.20}$$

$$= C_0 I_0 + \underbrace{C_1 I_{-1}}_{\epsilon/\epsilon}. \tag{8.2.21}$$

R is of UV origin only and is not seen by cuts in $4d$. By using generalized unitarity cuts we can project out the rational coefficients (C_B, C_T, C_b) form the factorised loop integrand, not computing integrals. For example, representing the cut with the line l_i $(i = 1, 2, 3, 4)$, so putting all the internal propagator on-shell simultaneously:

$$C_B = \frac{1}{2} \sum_{\substack{\text{solution} \\ \text{of } l_i^2 = 0}} \sum_{h_i = \pm} \tag{8.2.22}$$



Anatomy of (8.2.22): We are cutting the diagram in 4 parts, using the cuts l_i, with $i = 1, 2, 3, 4$, so we're distinguishing 4 sub-amplitude at tree-level. We've said that we have 4 degree of freedom for the loop-integral, when we impose (8.2.22), so 4 condition, we have everything fixed, and there is no integration to do in the quardupole cut case.

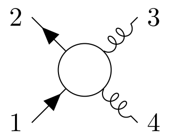
Below the summation sign in equation (8.2.22), you can see the text *solution of* $l_i^2 = 0$. In order to simultaneously satisfy the condition that all 4 internal momenta are on-shell, so $l_i^2 = 0$ for the massless case, in 4 dimensions, we are forced to allow the particle momenta to take complex values. In this complex kinematics framework, there are exactly two discrete solutions for the loop momentum that satisfy the cuts, which are classified using the spinor helicity formalism (the secondo sum).

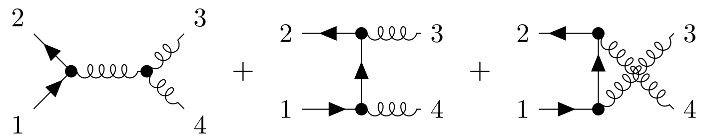
The arc serves as a visual indicator to show how the external momenta and the "cut" loop momenta combine at that specific vertex to satisfy one of the two mirror kinematic solutions. Depending on how the arc "brackets" or embraces the lines, physicists can immediately tell whether that specific vertex behaves as a 3-point or 4-point amplitude with specific helicity configurations (such as mostly plus or mostly minus helicities).

Observation. Systematic algorithms to proceed iteratively from the *maximal cut* to determine all coefficient, and supplying $d = 4 - 2\epsilon$ information to determine R, have been developed. Since there is no integration (purely algebraic) this algorithm can be implemented numerically.

8.3 Non-abelian gauge invariance: unitarity and ghost

In chapter §6 (see the section §6.1.2) we computed the tree-level amplitude for $q\bar{q}gg$ scattering (massless):

$\mathcal{A} =$  (8.3.1)

$=$  (8.3.2)

We showed that the Ward identity was satisfied:

$$\mathcal{A}^\mu p_{3\mu} = 0 \qquad \text{where} \quad \mathcal{A} = \mathcal{A}^\mu \epsilon_{3\mu} \tag{8.3.3}$$

there we will consider a tensor $\mathcal{M}^{\mu\nu}$ where $\mathcal{M}^{\mu\nu}\epsilon_{3\mu}\epsilon_{4\nu} = \mathcal{A}$ and $|\mathcal{A}|^2$ is related to $\mathcal{M}$ by the spin sum. We write this is a generic gauge:

$$\sum_s \epsilon_s^\mu(p,n)\epsilon_s^\nu(p,n) = -g^{\mu\nu} + P^{\mu\nu}(p,n) \tag{8.3.4}$$

with:

$$P_{\mu\nu}(p,n) = \frac{p^\mu n^\nu + p^\nu n^\mu}{p \cdot n} \qquad \text{where} \quad n^2 = 0. \tag{8.3.5}$$

Therefore:

$$\begin{aligned} |\mathcal{A}|^2 = {} & \mathcal{M}^{\mu_3\mu_4}(\mathcal{M}^{\nu_3\nu_4})^* \times \\ & \times \Big(- g_{\mu_3\nu_3} + P_{\mu_3\nu_3}(p_3,n)\Big)\Big(- g_{\mu_4\nu_4} + P_{\mu_4\nu_4}(p_4,n)\Big) \end{aligned} \tag{8.3.6}$$

naming respectively $1,2,3$ the three diagrams in (8.3.2). The contributions from diagrams 2 and 3 to $\mathcal{M}^{\mu\nu}$ are:

$$\begin{aligned} \mathcal{M}_{23}^{\mu_3\mu_4} = {} & -ig_s^2\overline{u}_1\gamma^{\mu_4}\frac{\not{p}_{23}}{s_{23}}\gamma^{\mu_3}v_2(t^{a_3}t^{a_4})_{i_2i_1} - \\ & - ig_s^2\overline{u}_1\gamma^{\mu_3}\frac{\not{p}_{24}}{s_{24}}\gamma^{\mu_4}v_2(t^{a_4}t^{a_3})_{i_2i_1}. \end{aligned} \tag{8.3.7}$$

Let's consider the contraction $\mathcal{M}^{\mu_3\mu_4}p_{3\mu}$ (instead of $\mathcal{M}^{\mu_3\mu_4}p_{3\mu_3}\epsilon_{4\mu_4}$ as we did before):

$$\begin{aligned} \mathcal{M}_{23}^{\mu_3\mu_4}p_{3\mu} = {} & -ig_s^2\overline{u}_1\gamma^{\mu_4}\frac{\not{p}_{23}}{s_{23}}\not{p}_3 v_2(t^{a_3}t^{a_4})_{i_2i_1} - \\ & - ig_s^2\overline{u}_1\not{p}_3\frac{\not{p}_{24}}{s_{24}}\gamma^{\mu_4}v_2(t^{a_4}t^{a_3})_{i_2i_1} \end{aligned} \tag{8.3.8}$$

$$= -ig_s^2\overline{u}_1\gamma^{\mu_4}v_2[t^{a_3},t^{a_4}]_{i_2i_1}. \tag{8.3.9}$$

The contribution from diagram 1 is instead:

$$
\begin{aligned}
\mathcal{M}_1^{\mu_3\mu_4} p_{3\mu} &= i g_s^2 \overline{u}_1 \gamma^\mu v_2 \frac{1}{s_{12}} [t^{a_3}, t^{a_4}]_{i_2 i_1} \Big(p_2^{\mu_4} (p_3 - p_4)_\mu + \\
&\qquad + g^{\mu_4}{}_\mu 2 p_3 \cdot p_4 + p_{3\mu} (-2p_3 - p_4)^{\mu_4} \Big) && (8.3.10) \\
&= i g_s^2 \overline{u}_1 \gamma^\mu v_2 \frac{1}{s_{12}} \Big(-p_3^{\mu_4} (p_3 + p_4)_\mu + g^{\mu_4}{}_\mu s_{12} - p_{3\mu} p_4^{\mu_4} \Big) [t^{a_3}, t^{a_4}]_{i_2 i_1} && (8.3.11) \\
&= i g_s^2 \Big(\overline{u}_2 \gamma^{\mu_4} v_2 - \overline{u}_1 \not{p}_3 v_2 p_4^{\mu_4} \Big) [t^{a_3}, t^{a_4}]_{i_2 i_1} && (8.3.12)
\end{aligned}
$$

so, using the fact that the second term in the right side vanishes when contracted with $\epsilon_{4\mu_4}$, we get:

$$
\mathcal{M}_1^{\mu_3\mu_4} p_{3\mu} = i g_s^2 \overline{u}_2 \gamma^{\mu_4} v_2 \epsilon_{4\mu_4} [t^{a_3}, t^{a_4}]_{i_2 i_1}. \tag{8.3.13}
$$

Before jump into a new section let's see some observations. First of all, in QED (an abelian gauge theory) the color factor is not present, so effectively:

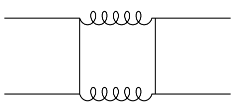
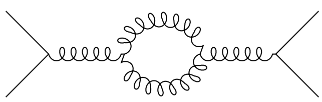
$$
[t^{a_3}, t^{a_4}]_{i_2 i_1} = 0. \tag{8.3.14}
$$

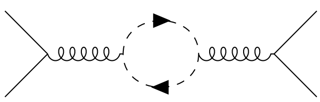
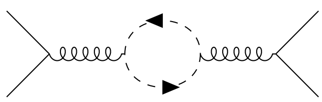
We can also see that what we have found is not in conflict with the Ward identity.

Also, there are implications for the optical theorem when connecting branch cuts of loop amplitudes to $|\mathcal{A}|^2$. Indeed, the optical theorem states that we must sum over intemediate states:

$$
2i\,\mathrm{Im}\{\mathcal{A}(q\bar{q} \to q\bar{q})\} = \sum_k \int \mathrm{d}\Phi_k\, |\mathcal{A}(q\bar{q} \to k)|^2. \tag{8.3.15}
$$

In QCD this means we must also consider closed ghost loops and gluon loops on the LHS:

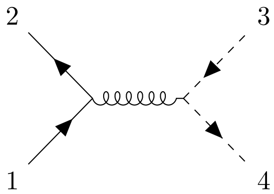
$\mathcal{A}^{(1)}(q\bar{q} \to q\bar{q}) =$  $+ \ldots +$  $+$

$+$  $+$ 

(8.3.16)

To see this is all consistent with our computation of $\mathcal{M}^{\mu_3\mu_4}$ we may also look at another amplitude:

$$\mathcal{A}_{ghost}(3,4) = \tag{8.3.17}$$



$$= -ig_s^2 \overline{u}_1 \gamma^\mu v_2 \frac{1}{s_{12}} p_{3\mu} [t^{a_4}, t^{a_3}]_{i_2 i_1} \tag{8.3.18}$$

$$= -ig_s^2 \overline{u}_1 \not{p}_3 v_2 \frac{1}{s_{12}} [t^{a_4}, t^{a_3}]_{i_2 i_1}. \tag{8.3.19}$$

This is very similar to the result obtained for:

$$\mathcal{M}^{\mu_3\mu_4} p_{3\mu_3} = p_4^{\mu_4} \mathcal{A}_{ghost}(3,4). \tag{8.3.20}$$

Swapping $3 \leftrightarrow 4$ give another relation:

$$\mathcal{M}^{\mu_3\mu_4} p_{4\mu_4} = p_3^{\mu_3} \mathcal{A}_{ghost}(4,3) \tag{8.3.21}$$

that we can use when expanding the tree amplitude squared using (8.3.6):

$$|\mathcal{A}|^2 = \mathcal{M}^{\mu_3\mu_4} (\mathcal{M}^{\nu_3\nu_4})^* \times \tag{8.3.22}$$

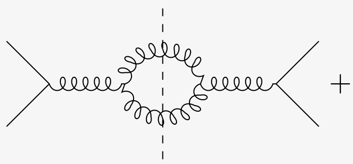
$$\times \Big(-g_{\mu_3\nu_3} + \frac{P_{3\mu_3} n_{\nu_3} + P_{3\nu_3} n_{\mu_3}}{P_3 \cdot n} \Big) \times$$

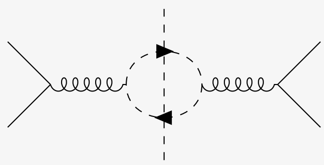
$$\times \Big(-g_{\mu_4\nu_4} + \frac{P_{4\mu_4} n_{\nu_4} + P_{4\nu_4} n_{\mu_4}}{P_4 \cdot n} \Big). \tag{8.3.23}$$

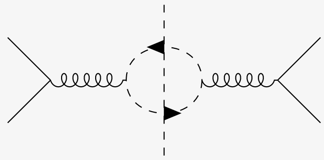
Exercise 19. Show that:

$$(8.3.23) = |\mathcal{M}|^2 + |\mathcal{A}_{ghost}(3,4)|^2 + |\mathcal{A}_{ghost}(4,3)|^2$$

$$= \left(\begin{array}{c}\text{squared amplitude}\\ \text{using Feynman gauge}\\ \text{propagator}\end{array}\right) + \left(\begin{array}{c}\text{ghost loop}\\ \text{in one}\\ \text{direction}\end{array}\right) + \left(\begin{array}{c}\text{ghost loop}\\ \text{in opposite}\\ \text{direction}\end{array}\right)$$

$=$  $+$

$+$ 

$+$ 

So we see that the ghost contributions are essential to mantain unitarity when working in the Feynman gauge. Note that if we work in the *light-like axial gauge*, where the spin sum matches the Feynmn rule for the propagator, unitarity is maintained without needing ghosts (but, as we noticed talking about the QCD lagrangian, this is only a specific gauge choice).

8.3.1 BRST symmetry

A complete understanding of non-abelian gauge invariance, unitarity and the role of ghost fields may be found using the formalism of *Becchi, Rouet, Stora and Tyutin* (BRST) ['74, '76].

The issue is that, while the classial QCD lagrangian is invariant under $SU(N_c)$ rotation:

$$\mathcal{L}_{QCD}^{cl} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \overline{\psi}(i\not{D} - m)\psi \tag{8.3.24}$$

using 1 quark flavour, quantization of the gauge field introduces new elements to the lagrangian:

$$\mathcal{L}_{QCD}^{gf} = -\frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 \tag{8.3.25}$$

$$\mathcal{L}_{QCD}^{ghost} = -\overline{c}^a \partial_\mu \mathcal{D}_{ab}^\mu c^b \tag{8.3.26}$$

for (covariant) gauge fixing and ghost respectively. These terms breaks the invariance of the lagrangian under $SU(N_c)$ rotations. There is however a hidden symmetry that we can uncover through the addition of an auxillary scalar field B^a. Let's compare two lagrangians:

$$\mathcal{L}_{QCD} = \mathcal{L}_{QCD}^{cl} + \mathcal{L}_{QCD}^{ghost} + \mathcal{L}_{QCD}^{gf} \tag{8.3.27}$$

$$\mathcal{L}_{QCD}^{B} = \mathcal{L}_{QCD}^{cl} + \mathcal{L}_{QCD}^{ghost} + \frac{\xi}{2}(B^a)^2 + B^a \partial_\mu A^{a\mu} \tag{8.3.28}$$

$$= \mathcal{L}_{QCD}^{cl} + \mathcal{L}_{QCD}^{ghost} + \mathcal{L}_{QCD}^{aux}. \tag{8.3.29}$$

Derivatives of the field B^a do not appear in $\mathcal{L}_{QCD}^{aux}$ hence it is a *non-dynamical field*. We may integrate this field out of the lagrangian using a generating function:

$$\int \mathcal{D}_B \, \exp\left\{ i \int \mathrm{d}^4x \, \mathcal{L}_{QCD}^{aux} \right\} = \tag{8.3.30}$$

$$= \int \mathcal{D}_B \, \exp\left\{ i\frac{\xi}{2} \int \mathrm{d}^4x \, \left(B^a + \frac{1}{\xi}\partial_\mu A^{a\mu} \right)^2 - \frac{1}{\xi^2}(\partial_\mu A^{a\mu})^2 \right\} \tag{8.3.31}$$

$$= \exp\left\{ i \int \mathrm{d}^4x \, \left(-\frac{1}{2\xi}(\partial_\mu A^{a\mu}) \right) \right\} \int \mathcal{D}_B \, \exp\left\{ i \int \mathrm{d}^4x \, (B^a)^2 \right\} \tag{8.3.32}$$

$$= \mathcal{N} \exp\left\{ i \int \mathrm{d}^4x \, \left(-\frac{1}{2\xi}(\partial_\mu A^{a\mu}) \right) \right\} \tag{8.3.33}$$

where we put the normalisation:

$$\mathcal{N} = \int \mathcal{D}_B \exp\left\{ i \int \mathrm{d}^4 x \, (B^a)^2 \right\} \tag{8.3.34}$$

that will cancel for any correlation function, so we may say:

$$\mathcal{L}^B_{QCD} = \mathcal{L}_{QCD} \tag{8.3.35}$$

since they produce identical physics. Recalling correlation functions in a scalar theory:

$$\langle 0 | \, T\{\phi_1 \dots \phi_n\} \, | 0 \rangle = \lim_{T \to \infty} \frac{\int \mathcal{D}_\phi \, \phi_1 \dots \phi_n \exp\{ i \int \mathrm{d}^4 x \, \mathcal{L} \}}{\int \mathcal{D}_\phi \, \exp\{ i \int \mathrm{d}^4 x \, \mathcal{L} \}} \tag{8.3.36}$$

and noticing that $\mathcal{L}^B_{QCD}$ is symmetric under a global tranformation, we can write the tranformation, using an anti-commuting Grassmann parameter θ:

$$\delta_\theta A^{a\mu} = \theta \mathcal{D}^{ab}_\mu c^b \tag{8.3.37}$$
$$\delta_\theta \psi = i g_s \theta c^a t^a \psi \tag{8.3.38}$$
$$\delta_\theta c^a = -\frac{1}{2} g_s \theta f^{abc} c^b c^c \tag{8.3.39}$$
$$\delta_\theta \overline{c}^a = \theta B^a \tag{8.3.40}$$
$$\delta_\theta B^a = 0. \tag{8.3.41}$$

$\delta A^{a\mu}$ and $\delta\psi$ are equivalent to infinitesimal $SU(N_c)$ rotation through an angle θ_{SU}:

$$\Theta = \theta^a_{SU}(x) t^a \tag{8.3.42}$$
$$= \theta c^a(x) t^a \tag{8.3.43}$$

and, recalling that $SU(N_c)$ leads to:

$$A'_\mu = A_\mu - \mathcal{D}_\mu \Theta + \mathcal{O}(\Theta^2) \tag{8.3.44}$$
$$\psi' = \exp\{ i g_s \Theta \} \psi = (1 + i g_s \Theta)\psi + \mathcal{O}(\Theta^2) \tag{8.3.45}$$

we have the correct sign, since θ anticommutes with c^a:

$$\theta c^a(x) = -c^a(x)\theta. \tag{8.3.46}$$

We may now show that:

$$\delta_\theta(\mathcal{L}_{QCD}) = \delta_\theta(\mathcal{L}^B_{QCD}) = 0. \tag{8.3.47}$$

We have:

$$\mathcal{L}^B_{QCD} = \mathcal{L}^{cl}_{QCD} + \mathcal{L}^{ghost}_{QCD} + \mathcal{L}^{aux}_{QCD} \tag{8.3.48}$$

and we can se the variation (for $SU(N_c)$ rotation):

$$\delta_\theta(\mathcal{L}^{cl}_{QCD}) = 0 \tag{8.3.49}$$

$$\delta_\theta(\mathcal{L}^{aux}_{QCD}) = \delta_\theta\left(\frac{\xi}{2}(B^a)^2 + B^a\partial_\mu A^{a\mu}\right) \tag{8.3.50}$$

$$= \frac{\xi}{2}\underbrace{\delta_\theta\left[(B^a)^2\right]}_{=0} + \underbrace{\delta_\theta(B^a)}_{=0}\partial_\mu A^{a\mu} + B^a\partial_\mu\delta_\theta(A^{a\mu}) \tag{8.3.51}$$

$$= B^a\partial^\mu(\theta\mathcal{D}^{ab}_\mu c^b) \tag{8.3.52}$$

and also:

$$\delta_\theta(\mathcal{L}^{ghost}_{QCD}) = \delta_\theta(-\bar{c}^a\partial_\mu\mathcal{D}^{\mu ab}c^b) \tag{8.3.53}$$

$$= -\delta_\theta(\bar{c}^a)\partial_\mu\mathcal{D}^{\mu ab}c^b - \bar{c}^a\partial_\mu\delta_\theta\left(\partial^\mu c^a + g_s f^{abc}A^{\mu c}c^b\right) \tag{8.3.54}$$

$$\begin{aligned}= &-\theta B^a\partial_\mu\mathcal{D}^{\mu ab}c^b - \bar{c}^a\partial_\mu\partial^\mu\left(-\frac{1}{2}\theta f^{abc}c^bc^c\right) - \\ &- \bar{c}^a\partial_\mu\left[g_s f^{abc}\Big((\delta_\theta A^{\mu c})c^b + A^{\mu c}\delta_\theta(c^b)\Big)\right]\end{aligned} \tag{8.3.55}$$

$$\begin{aligned}= &-\theta B^a\partial_\mu\mathcal{D}^{\mu ab}c^b - \bar{c}^a\partial_\mu\partial^\mu\left(-\frac{1}{2}g_s\theta f^{abc}c^bc^c\right) - \\ &- g_s f^{abc}\bar{c}^a\partial_\mu\left[\theta(\mathcal{D}^{\mu cd}c^d)c^b + A^{\mu c}\left(-\frac{1}{2}g_s\theta f^{bde}c^dc^e\right)\right]\end{aligned} \tag{8.3.56}$$

$$\begin{aligned}= &-\theta B^a\partial_\mu\mathcal{D}^{\mu ab}c^b + g_s\bar{c}^a\partial_\mu\theta f^{abc}\frac{1}{2}\partial^\mu(c^bc^c) - \\ &- g_s f^{abc}\bar{c}^a\theta\partial_\mu\left[(\partial^\mu c^c)c^b + g_s f^{cde}A^{\mu e}c^dc^b - \frac{1}{2}g_s f^{bde}A^{\mu c}c^dc^e\right].\end{aligned} \tag{8.3.57}$$

Exercise 20. Collect terms at $\mathcal{O}(g_s)$ and $\mathcal{O}(g_s^2)$ to show that $\mathcal{O}(g_s)$ and $\mathcal{O}(g_s^2)$ cancel (with help from the Jacobi identity for the latter).

You can see the solution for the exercises in the Appendix B; this leaves:

$$\delta_\theta(\mathcal{L}^{ghost}_{QCD}) = -\theta B^a\partial_\mu\mathcal{D}^{\mu ab}c^b = -\delta_\theta(\mathcal{L}^{aux}_{QCD}) \tag{8.3.58}$$

and therefore:

$$\delta_\theta(\mathcal{L}_{QCD}) = 0. \tag{8.3.59}$$

This global symmetry implies a conserved (Noether) current:

$$\partial_\mu j^\mu_{BRST} = 0 \tag{8.3.60}$$

from which we may also take a conserved charge:

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_{BRST} = 0 \tag{8.3.61}$$

So we have:

$$Q_{BRST} = \int \mathrm{d}^3\vec{x}\, j^0_{BRST} \tag{8.3.62}$$

and the BRST charge commutes with the Hamiltonian:

$$[Q_{BRST}, H] = 0. \tag{8.3.63}$$

The variation of the gauge field (8.3.37) can be written in terms of the BRST charge:

$$\delta_\theta A^a_\mu = \theta\big[Q_{BRST}, A^a_\mu\big] \tag{8.3.64}$$

and similarly for other fields $(B, c, \bar{c}, \psi)$. We can say this with a simpler notation:

$$\delta_\theta F = \theta Q F. \tag{8.3.65}$$

Exercise 21. Show that $Q^2 F = 0$ for a field:

$$F = (A^a_\mu, \psi, c^a, \bar{c}^a, B^a). \tag{8.3.66}$$

This exercise shows the Q^2 is a *nilpotent* operator and we may write:

$$Q^2 = 0. \tag{8.3.67}$$

8.3.2 Implications of BRST symmetry for unitarity

A nilpotent operator that commutes with the hamiltonian will divide the Hilbert space, $\mathcal{H}$, into 3 subspace:

- $\mathcal{H}_0$ with states $|\psi_0\rangle$ where $Q|\psi_0\rangle = 0$ and $\nexists\, |\phi\rangle \in \mathcal{H}$ satisfying $|\psi_0\rangle = Q|\phi\rangle$.
- $\mathcal{H}_1$ with states $|\psi_1\rangle$ where $Q|\psi_1\rangle \neq 0$.
- $\mathcal{H}_2$ with states $|\psi_2\rangle$ where $|\psi_2\rangle = Q|\phi\rangle$ (with $|\phi\rangle \in \mathcal{H}$) such that $Q|\psi_2\rangle = Q^2|\phi\rangle = 0$.

This defines what we call the **BRST cohomology**. While we do not explore this further it is the first hint of a deep mathematical structure.

We can do some observation. We can see that the states of $\mathcal{H}_2$ are orthogonal: for $|\psi_2\rangle, |\psi_2'\rangle \in \mathcal{H}_2$ we can see:

$$\langle\psi_2|\psi_2'\rangle = \langle\phi Q^\dagger Q\phi'\rangle = \langle\phi Q^2\phi'\rangle = 0. \tag{8.3.68}$$

We can see that the states of $\mathcal{H}_2$ are orthogonal to those of $\mathcal{H}_0$:

$$\langle\psi_2|\psi_0\rangle = \langle\psi_2| Q^\dagger |\psi_0\rangle = \langle\psi_2| Q |\psi_0\rangle = 0. \tag{8.3.69}$$

We can interpret the structure as follows:

- $\mathcal{H}_0$ contains physical states.
- $\mathcal{H}_1$ and $\mathcal{H}_2$ contain unphysical states, so the ghosts and longitudinal polarisations.

Using the BRST charge and the states of each subspace we may prove the unitarity of the S-matrix:

$$[Q_{BRST}, H] = 0 \quad \longleftrightarrow \quad [Q, S] = 0. \tag{8.3.70}$$

Therefore, for $|\psi_0\rangle$ in $\mathcal{H}_0$:

$$0 = [Q, S] |\psi_0\rangle \tag{8.3.71}$$
$$= (QS - SQ) |\psi_0\rangle \tag{8.3.72}$$
$$= QS |\psi_0\rangle \tag{8.3.73}$$

using $Q|\psi_0\rangle = 0$. This means that the states $S|\psi_0\rangle$ must be in $\mathcal{H}_0$ or in $\mathcal{H}_2$ and not in $\mathcal{H}_1$:

$$S|\psi_0\rangle \in \mathcal{H}_0 \oplus \mathcal{H}_2. \tag{8.3.74}$$

Now we look at two physical states from $\mathcal{H}_0$, $|\psi_0\rangle$ and $|\psi_0'\rangle$, where:

$$\langle\psi_0'|\psi_0\rangle = \langle\psi_0'| S^\dagger S |\psi_0\rangle \tag{8.3.75}$$

using $S^\dagger S = 1$. We can insert a complete set of states with:

$$\sum_{\psi\in\mathcal{H}} |\psi\rangle\langle\psi| = \mathbb{1} \tag{8.3.76}$$

in this way we get:

$$\langle\psi_0| S^\dagger S |\psi_0\rangle = \sum_{\psi\in\mathcal{H}} \langle\psi_0'| S^\dagger |\psi\rangle\langle\psi| S |\psi_0\rangle \tag{8.3.77}$$

$$\begin{aligned} &= \sum_{\psi''\in\mathcal{H}_0} \langle\psi_0'| S^\dagger |\psi_0''\rangle\langle\psi_0''| S |\psi_0\rangle + \\ &\quad + \sum_{\psi_1\in\mathcal{H}_1} \langle\psi_0'| S^\dagger |\psi_1\rangle \underbrace{\langle\psi_1| S |\psi_0\rangle} + \\ &\quad + \sum_{\psi_2\in\mathcal{H}_2} \langle\psi_0'| S^\dagger |\psi_2\rangle \underbrace{\langle\psi_2| S |\psi_0\rangle} \end{aligned} \tag{8.3.78}$$

where we can say that the two underlined terms are equals to zero because $\mathcal{H}_1$ is orthogonal to $\mathcal{H}_0 \oplus \mathcal{H}_2$ and $\mathcal{H}_2$ is orthogonal to $\mathcal{H}_1$ and $\mathcal{H}_2$.

So, finally, we can conclude that the S-matrix elements for physical states is unitary.

8.4 A bierf recap

I wanted to make in this section a brief recap of what we saw in this chapter to understand better what we've done. It's a personal section, so you can skip it an go to §8.5.

The scattering operator S must be unitary because total probability is conserved: starting from an initial state, the sum of the transition probabilities to *all* possible final states must add up to 1. Writing:

$$S = \mathbb{1} + iT, \tag{8.4.1}$$

the condition $SS^\dagger = \mathbb{1}$ becomes a relation between T and $T^\dagger$. Expanding in powers of the coupling constant g one finds a hierarchy of relations between amplitudes at different orders in perturbation theory: the tree-level amplitude is real, while at one loop its imaginary part (discontinuity) is tied to a sum over intermediate states of products of lower-order amplitudes.

The amplitude $\mathcal{M}(s)$ should be thought of as a complex analytic function of the center-of-mass energy $s = E_{\text{cm}}^2$. Below the production threshold ($s < s_0$) no intermediate state can be real (on-shell): all propagators stay nonzero and $\mathcal{M}(s)$ is real, i.e.:

$$\mathcal{M}(s) = [\mathcal{M}(s^*)]^* \,. \tag{8.4.2}$$

Above threshold ($s > s_0$) this is no longer true: analytically continuing to $s \pm i\epsilon$ one finds a *branch cut* on the real axis, with discontinuity:

$$\mathrm{Disc}_s\,\mathcal{M} \equiv \mathcal{M}(s+i\epsilon) - \mathcal{M}(s-i\epsilon) = 2i\,\mathrm{Im}\,\mathcal{M}(s+i\epsilon). \tag{8.4.3}$$

A simple example that clarifies the origin of the cut is the logarithm.

We've saw a worked example: the bubble in φ^4 theory. Consider the simplest possible diagram: the one-loop bubble I_2 in the s_{12} channel of the $2 \to 2$ amplitude.

The loop integral has poles in the complex k_0 plane (energy of the loop momentum), corresponding to the propagator going on-shell. As long as the energy in the channel is below threshold ($p_0 < 2m$), these poles do not sit on the integration contour: the integral is real, no discontinuity. Above threshold ($p_0 > 2m$) a pole crosses the integration contour: a discontinuity arises, computable via the Sokhotski–Plemelj formula:

$$\frac{1}{x \pm i\epsilon} = P\left(\frac{1}{x}\right) \mp i\pi\,\delta(x). \tag{8.4.4}$$

The practical result is that the discontinuity is obtained by replacing each "cut" propagator with an on-shell Dirac delta:

$$\frac{1}{k^2 - m^2 + i\epsilon} \;\longrightarrow\; -2\pi i\,\delta^{(+)}(k^2 - m^2). \tag{8.4.5}$$

This is the *Cutkosky rule.*

But, why it works: the physical meaning of the cut. A virtual propagator

$$\frac{1}{k^2 - m^2 + i\epsilon} \tag{8.4.6}$$

is in general a complex number. Splitting it via Sokhotski–Plemelj:

$$\frac{1}{x + i\epsilon} = P\left(\frac{1}{x}\right) - i\pi\,\delta(x), \tag{8.4.7}$$

the first piece (principal value) exists even for virtual particles off the mass shell; the second (the delta) exists only if $k^2 = m^2$ is reachable within the integration domain, i.e. only if the virtual particle can become *real and on-shell.*

When one computes the discontinuity of the amplitude, the principal-value contributions cancel between $+i\epsilon$ and $-i\epsilon$, and only the delta term survives. "Cutting" a propagator therefore means *treating that internal particle as real and on-shell*, rather than virtual. The loop integral thus turns into a phase-space integral over the real particles produced as an intermediate state — the very same quantity that appears in the optical theorem:

$$2\,\mathrm{Im}(\mathcal{M}) = \sum_{\substack{\text{intermediate}\\ \text{states}}} \int d\Pi\,|\mathcal{M}|^2. \tag{8.4.8}$$

Generalizing the previous example, the Cutkosky rules are:

1. Cut the diagram in every possible way such that the cut propagators can simultaneously be put on-shell.
2. For each cut propagator, replace
$$\frac{1}{k^2 - m^2 + i\epsilon} \longrightarrow -2\pi i\,\delta^{(+)}(k^2 - m^2), \tag{8.4.9}$$
and integrate.
3. Sum over all possible cuts.

The physical meaning: the discontinuity of the loop amplitude is given by the sum, over all possible intermediate states, of the product of the amplitude to the left of the cut and the complex conjugate of the amplitude to the right — exactly the *optical theorem.*

When an internal propagator goes on-shell, the amplitude factorizes into the product of two simpler (lower-order) amplitudes, joined by the on-shell particle:

$$\lim_{s_{12}\to 0} \mathcal{A}(\ldots) \sim \frac{i}{s_{12}}\,\mathcal{A}_L^{\text{tree}}\,\mathcal{A}_R^{\text{tree}}. \tag{8.4.10}$$

In cases with spin this works because the sum over physical polarization states reproduces the propagator (up to gauge terms):

$$\frac{p^\mu\gamma_\mu + m}{p^2 - m^2} \longrightarrow \sum_s u_s\bar{u}_s, \qquad \frac{-g^{\mu\nu} + p^\mu n^\nu + n^\mu p^\nu/(p\cdot n)}{p^2 - m^2} \longrightarrow \sum_n \varepsilon_n^\mu \varepsilon_n^{*\nu}. \tag{8.4.11}$$

Applying *several cuts at once* (not just one channel, but several propagators simultaneously) is called *generalized unitarity*, and it is a powerful tool for reconstructing loop amplitudes without ever computing the loop integral explicitly.

But why the cuts guarantees the unitarity?

Unitarity $SS^\dagger = \mathbb{1}$, written in terms of T for a forward amplitude (initial state = final state), is the optical theorem:

$$2\,\mathrm{Im}\,\mathcal{M}(i\to i) = \sum_f \int d\Pi_f\,|\mathcal{M}(i\to f)|^2. \tag{8.4.12}$$

That is: the imaginary part of the forward amplitude must equal exactly the sum, over all physical intermediate states f, of the squared modulus of the production amplitude, integrated over phase space. When computing amplitudes via Feynman diagrams it is by no means obvious that this equality holds: each diagram has real and imaginary parts arising from independent analytic calculations. The Cutkosky rules provide the proof, diagram by diagram and order by order, that the equality is indeed true.

Indeed, the logical chain is: the discontinuity of an amplitude is computed by replacing cut propagators with on-shell deltas (Cutkosky rule); an on-shell delta $\delta^{(+)}(k^2 - m^2)$ with positive energy is exactly the factor that appears in the phase-space measure $d\Pi$ of a real physical particle; cutting a loop diagram in two automatically yields: (tree-level amplitude to the left of the cut) $\times$ (complex conjugate of the amplitude to the right) $\times$ (phase-space measure of the cut state) — i.e. exactly one term of the sum $\sum_f \int d\Pi_f\,|\mathcal{M}|^2$; summing over all possible cuts of a diagram gives the sum over all intermediate states f allowed at that perturbative order.

Summing all cuts of all diagrams at a given order in g reconstructs, piece by piece, the right-hand side of the optical theorem.

The cutting rules do not impose unitarity from the outside: if one computes the discontinuity in two independent ways — once from the poles of the loop integral, and once from the product of tree-level amplitudes integrated over phase space — and the two results agree (as in the case of the bubble in φ^4, comparing (8.1.69) and (8.1.71)), one has an *explicit proof*, diagram by diagram, that the theory satisfies $SS^\dagger = \mathbb{1}$ at that perturbative order.

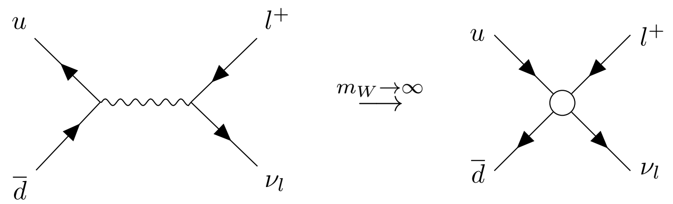
In non-abelian gauge theories, as usual, is a little bit different. Indeed, in non abelian gauge theories cutting an internal gluon propagator, summed over all physical polarizations, is *not enough* on its own to reproduce the optical theorem: the gluon propagator in covariant gauge also contains unphysical degrees of freedom (longitudinal and temporal polarizations). It is necessary to also include *ghost cuts* in the intermediate state, which exactly cancel the spurious contributions.

The cutting rules, applied correctly (including ghosts), are therefore also the tool used to prove that non-abelian gauge theories remain unitary despite the introduction of the ghosts themselves. This is the method used by 't Hooft and Veltman; the *largest time equation* is the systematic generalization of this argument to arbitrarily complicated diagrams, with any number of internal vertices.

8.5 Invitation: Effective field theories

We have stated that theories with operators of mass dimension higher than 4 are not renormalizable (in a $4d$ space-time). We may relax this condition if we only require our field theory to apply within a specific energy regime and define an *effective field theory*.

The Fermi theory of the Weak interaction is the most famous example:



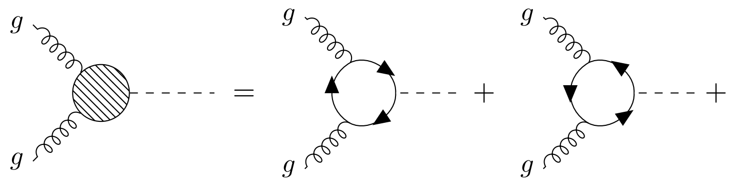
that means that the propagator:

$$\frac{1}{p^2 - m_W^2} \quad \xrightarrow{m_W \to \infty} \quad -\frac{1}{m_W^2}\left(1 + \frac{p^2}{m_W^2} + \dots\right) \tag{8.5.1}$$

where the term p^2/m_W^2 is the correction froms higher derivatives suppressed by $1/m_W^2$. You can see the complete description of the theory in MSc course *Phenomenology of Fundamental Interaction*'s lecture notes.

We can use an explicit computation of $gg \to H$ via a top quark loop to illustrate how this can work in practice. The process is loop induced in the

Standard Model (SM):



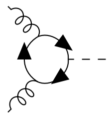
$$+\,\mathcal{O}\left(g_s^4 y_t, g_s^2 y_t^3, g_s^2 y_t g_W^2\right). \tag{8.5.2}$$

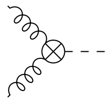
The relevant part of the SM is:

$$\mathcal{L}_{SM} = \dots\ y_t H \overline{\psi}_t \psi_t + \overline{\psi}_t (i\not{D} - m_t)\psi_t\ \dots \tag{8.5.3}$$

distinguishing between L and R handed quark fields is not relevant here (because are involved only strong interaction and Higgs-top coupling). In the region $\sqrt{s} \ll m_t/m_H$ we may use an approximation $m_t \to \infty$ (Higgs couples only the top quark):

$$\begin{array}{ccccc} \mathcal{L}_{SM} & \xrightarrow{m_t\to\infty} & \mathcal{L}_{SM}^{(\text{no top})} & + & \mathcal{L}_{eff} \\ \downarrow & & \downarrow & & \downarrow \\ \text{contains} & & \text{no Yukawa} & & \text{contains} \end{array} \tag{8.5.4}$$



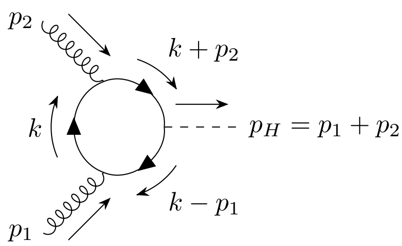


Since we have the technology to compute the loop diagram directly we may see how to write $\mathcal{L}_{eff}$ in terms of a dimension 5 operator. The two diagrams are symmetric so:

$$\mathcal{A}(gg \to H) = 2\mathcal{D} \tag{8.5.5}$$

where:

$\mathcal{D} =$



Since the process is loop induced the loop amplitude must be finite we have:

$$\mathcal{D} = (-ig_s)^2 \left(\frac{-iy_t}{\sqrt{2}}\right) \mathrm{Tr}\{t^{a_1} t^{a_2}\} \epsilon_{1\mu_1} \epsilon_{2\mu_2} I^{\mu_1\mu_2} \tag{8.5.6}$$

where the loop integral $I^{\mu_1\mu_2}$ is:

$$I^{\mu_1\mu_2} = \int_k \frac{N^{\mu_1\mu_2}}{D_1 D_2 D_3} \tag{8.5.7}$$

and we have written:

$$D_1 = D(k, m_t) \quad , \quad D_2 = D(k + p_2, m_t) \quad , \quad D_3 = D(k - p_1, m_t). \tag{8.5.8}$$

while the numerator is:

$$N^{\mu_1\mu_2} = \mathrm{Tr}\Big\{\not\epsilon_1(\not k + m_t)\not\epsilon_2(\not k + \not p_2 + m_t)(\not k - \not p_1 + m_t)\Big\}. \tag{8.5.9}$$

We will proceed dirctly to Feynman parameters and then perform tensor reduction. Writing:

$$\frac{1}{D_1 D_2 D_3} = 2\int_0^1 \mathrm{d}\alpha_1 \int_0^{1-\alpha_1} \mathrm{d}\alpha_2 \, \frac{1}{\Big[\alpha_1 D_2 + \alpha_2 D_3 + (1 - \alpha_1 - \alpha_2) D_1\Big]^3} \tag{8.5.10}$$

we may show:

$$\hat{D}(k) = \alpha_1 D_2 + \alpha_2 D_3 + (1 - \alpha_1 - \alpha_2) D_1 \tag{8.5.11}$$

$$= (k + \alpha_1 p_2 - \alpha_2 p_1)^2 + 2\alpha_1\alpha_2 p_1 \cdot p_2 - m_t^2 + i\emptyset^+ \tag{8.5.12}$$

where:

$$2p_1 \cdot p_2 = (p_1 + p_2)^2 = p_H^2 = m_H^2. \tag{8.5.13}$$

We may now proceed (dropping the term $i\emptyset^+$) to shift the loop momentum and evaluate the trace in the numerator $N^{\mu_1\mu_2}$.

Exercise 22. Show after Feynman parametrisation the diagram can be written:

$$I^{\mu\nu} = 2\int_k \int_0^1 \mathrm{d}\alpha_1 \int_0^{1-\alpha_1} \mathrm{d}\alpha_2 \, \frac{T^{\mu\nu}(k - \alpha_1 p_2 + \alpha_2 p_1)}{(k^2 - m_t^2 + \alpha_1\alpha_2 m_H^2)^3} \tag{8.5.14}$$

where:

$$\epsilon_{1\mu}\epsilon_{2\nu}T^{\mu\nu}(k) = 4m_t \left[g^{\mu\nu}\left(-k^2 - \frac{m_H^2}{2} - m_t^2\right) + 4k^\mu k^\nu + p_1^\nu p_2^\mu\right]\epsilon_{1\mu}\epsilon_{2\nu}. \tag{8.5.15}$$

You may use $\epsilon_i \cdot p_i = 0$.

For the tensor reduction in Feynman parameter space we make case of:

$$\int_k \frac{k^\mu k^\nu}{(k^2 - \Delta)^n} = \frac{1}{d} g^{\mu\nu} \int_k \frac{k^2}{(k^2 - \Delta)^n} \tag{8.5.16}$$

and:

$$\int \frac{k^\mu}{(k^2 - \Delta)^n} = 0. \tag{8.5.17}$$

Exercise 23. Show after reduction that:

$$I^{\mu\nu} = 2\int_k\int_\alpha \left[g^{\mu\nu}\left(\frac{d-4}{d}k^2 - m_t^2 + m_H^2(1-2\alpha_1\alpha_2)\right) - \right.$$
$$\left. -p_1^\nu p_2^\mu(1-4\alpha_1\alpha_2)\right] \frac{4m_t}{(k^2-m_t^2+m_H^2\alpha_1\alpha_2)^3}. \quad (8.5.18)$$

The next step is integration over k using the results:

$$\int_k \frac{1}{(k^2-\Delta)^3} = \frac{i}{(4\pi)^2}(4\pi)^\epsilon\Gamma(1+\epsilon)\frac{\Delta^{-1-\epsilon}}{2} \quad (8.5.19)$$
$$\int_k \frac{k^2}{(k^2-\Delta)^3} = \frac{i}{(4\pi)^2}(4\pi)^\epsilon\Gamma(1+\epsilon)\left(\frac{2-\epsilon}{\epsilon}\right)\Delta^{-\epsilon}. \quad (8.5.20)$$

NB. The $1/\epsilon$ pole in the k^2 integral is cancelled by the coefficient $d-4 = -2\epsilon$ in front; so the result is finite as expected.

Exercise 24. Perform loop momentum integration to show (recalling $y_t = m_t/v$):

$$i\mathcal{D} = -\frac{\alpha_s}{\pi}\frac{m_t^2}{v}\frac{\delta^{ab}}{2}\int_0^1 \mathrm{d}\alpha_1 \int_0^{1-\alpha_1} \mathrm{d}\alpha_2 \left(\frac{1-4\alpha_1\alpha_2}{m_t^2 - m_H^2\alpha_1\alpha_2}\right)\times$$
$$\times\left(-\epsilon_1\cdot\epsilon_2\frac{m_H^2}{2} + \epsilon_1\cdot p_2\,\epsilon_2\cdot p_1\right) + \mathcal{O}(\epsilon). \quad (8.5.21)$$

Exercise 25. Feynman paramater integration. Show that:

$$F\left(\frac{m_H^2}{m_t^2}\right) = \int_\alpha \frac{1-4\alpha_1\alpha_2}{\left(1-\frac{m_H^2}{m_t^2}\alpha_1\alpha_2\right)^3} \quad (8.5.22)$$

is a constant in the limit $m_t/m_H \to \infty$:

$$\lim_{x\to 0} F(x) = \frac{1}{3} + \mathcal{O}(x). \quad (8.5.23)$$

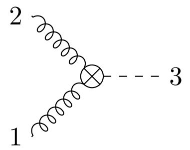
We have now completed the amplitude computation in the $m_t \to \infty$ (or more correctly $(m_t/m_H)^2 \to \infty$) limit:

$$\mathcal{A}(gg\to H) \quad \xrightarrow{\frac{m_t}{m_H}\to\infty} \quad \frac{\alpha_s}{3\pi v}\delta^{ab}(-g^{\mu\nu}p_1\cdot p_2 + p_1^\nu p_2^\mu)\epsilon_{1\mu}\epsilon_{2\nu} + \mathcal{O}\left(\frac{m_H^2}{m_t^2}\right). \quad (8.5.24)$$

The important feature of this result is that it has a local structure that can be matched to a local operator:

$$\mathcal{L}_{eff} = CH\,\mathrm{Tr}\big\{F^a_{\mu\nu}F^{a\mu\nu}\big\} \tag{8.5.25}$$

where C is known as a *Wilson coefficient* and H is the Higgs' field. The coefficient C can be found via a procedure called *matching*. From this lagrangian we find the Feynman rule:



$$= iC\left(-g^{\mu_1\mu_2}p_1\cdot p_2 + p_1^{\mu_2}p_2^{\mu_1}\right) \tag{8.5.26}$$

so matching with the result for $\mathcal{A}(gg\to H)$ is straightforward:

$$\begin{array}{ccc} \mathcal{L}_{SM} & & \mathcal{L}_{SM}^{(notop)} + \mathcal{L}_{eff} \\ \downarrow & & \downarrow \\ \mathcal{A}^{(1)}(gg\to H) & \overset{m_t/m_H\to\infty}{\longrightarrow} & \mathcal{A}^{(0)}_{eff}(gg\to H) + \mathcal{O}\left(\frac{m_H^2}{m_t^2}\right) \end{array} \tag{8.5.27}$$

$$\Longrightarrow \quad C = \frac{\alpha_s}{3\pi v} + \mathcal{O}(\alpha_s^2). \tag{8.5.28}$$

In general we may construct the set of higher dimension operator satisfying symmetry constraints (e.g. $SU(3)_c \times SU(2)_L \times U(1)_Y$, called $SMEFT$):

$$\mathcal{L}_{eff} = \mathcal{L}_0 + \sum_{k=1}^{\infty}\sum_m \frac{C_m^{(k+4)}}{\Lambda^k}O_m^{(k+4)} \tag{8.5.29}$$

where $C_m^{(d)}$ are Wilson coefficient of mass dimension d operators $O_m^{(d)}$. The sum over elements, m, must be done carefully using a basis of independent operators which accounts for field redefinitions, integration-by-parts identifies and equation of motion. Independent operator basis may be quite large. For the SM, with general flavour structure, we easily reach > 2000 operators.

Remark. In the full Standard Model, the Higgs boson (H) cannot couple directly to gluons because gluons are massless gauge bosons. Instead, the interaction occurs at the one-loop level through a triangle diagram containing heavy virtual quarks (predominantly top due to its large Yukawa coupling).

The key to transitioning to an *Effective Field Theory (EFT)* framework lies in comparing the mass scales of the problem: the mass of the Higgs boson is $m_H \approx 125$ GeV, while the mass of the top quark is significantly larger, at $m_t \approx 173$ GeV. Because $m_t \gg m_H$, we can study this process in the *large top-mass limit* ($m_t \to \infty$). At the energy scale of Higgs production, we do not have enough energy to produce real, on-shell top quarks; they only exist as highly virtual fluctuations inside the loop.

Mathematically, this allows us to perform a Taylor expansion of the loop integrand in powers of q^2/m_t^2 (where q is the external momentum). In configuration space, this expansion effectively shrinks the non-local quark triangle loop down into a single spacetime point.

The heavy top quark is "integrated out" of the theory, and its quantum effects are neatly packaged into the *Wilson coefficient* C. As a result, the non-local loop interaction collapses into a *higher-dimensional local operator*:

$$\mathcal{L}_{\text{eff}} = CH\,\text{Tr}\big\{F^a_{\mu\nu}F^{a\mu\nu}\big\} \tag{8.5.30}$$

This is a dimension-5 operator (which is a higher-dimensional operator, since the standard renormalizable Lagrangian only contains operators up to dimension 4). Even though the underlying physics involves a complex loop, the effective theory describes it as a simple, *"squeezed" (local) contact interaction* between the Higgs field and two gluon field-strength tensors.

Appendix

Appendix A

Natural Units

In this section, we study what is meant by natural units. In Physics, it is very common to use *God-given units*, i.e., the so-called natural units. The independent types of quantities are length $[L]$, time $[T]$, mass $[M]$, and — if we also consider thermodynamics and statistical mechanics — temperature $[T]$.

In nature, there are universal dimensional constants, such as the speed of light c, the reduced Planck constant $\hbar$, Newton's gravitational constant G_N, and the Boltzmann constant k. Typically, natural units are defined as those in which we set:

$$\hbar = c = 1 \tag{A.0.1}$$

and consequently, when modifying the quantities (in standard units):

$$\begin{cases} \hbar = 6.66 \cdot 10^{-22}\ \text{MeV} \cdot \text{s} \\ c = 3 \cdot 10^{8}\ \text{m/s} \\ \hbar c \sim 200\ \text{MeV} \cdot \text{fm} \end{cases} \quad \Longrightarrow \quad \begin{cases} \hbar = 1 \\ c = 1 \\ \hbar c = 1. \end{cases}$$

Stating that the speed of light is a dimensionless unit quantity implies that the units of length and time are equivalent:

$$[L] = [T] \tag{A.0.2}$$

consequently, we also have:[1]

$$[E] = [\vec{p}] = [M] \tag{A.0.4}$$

For a force F, we have:

$$[F] = [EL^{-1}] = [ML^{-1}] \tag{A.0.5}$$

[1]As a result of the relationship between mass and energy in Special Relativity:

$$E^2 = p^2 + m^2 \tag{A.0.3}$$

therefore, a quantity like $\hbar$, which has the dimensions of an action, i.e., $[ET]$, in natural units is:

$$[\hbar] = [ET] = [ML] = 1 \quad \Longrightarrow \quad [M] = [L^{-1}]. \tag{A.0.6}$$

From the relations observed between the quantities, only one of them is truly independent. Choosing length as the independent one, it holds:

$$[L] = [T] \quad , \quad [E] = [\vec{p}] = [M] = [L^{-1}] \quad , \quad [F] = [L^{-2}]. \tag{A.0.7}$$

Typically, in QFT, cross-sections are expressed in natural units, but this is not very convenient to use in the physical world, as we require standard units. In general (not only for cross-sections), by remembering the units of measurement of $\hbar$ and c, it is possible to convert a quantity from natural units to S.I. units by multiplying by appropriate powers of $\hbar$ and c. We have:

$$[\hbar] = ML^2T^{-1} \qquad ; \qquad [c] = LT^{-1}.$$

Let's look at an example. Take the Thomson cross-section in natural units and convert it to standard units. We have the process:

$$\gamma + e^- \longrightarrow \gamma + e^-$$

with the free e^- and the cross-section:

$$\sigma_T = \frac{8}{3}\pi\frac{\alpha^2}{m_e^2} \tag{A.0.8}$$

which we rewrite as:

$$\sigma_T = \frac{8}{3}\pi\frac{\alpha^2}{m_e^2}\hbar^x c^y \tag{A.0.9}$$

By analyzing the dimensions:

$$\begin{aligned} L^2 &= M^{-2}\left(ML^2T^{-1}\right)^x \left(LT^{-1}\right)^y \\ L^2 &= M^{x-2}L^{2x+y}T^{-x-y}. \end{aligned}$$

We must therefore solve the system:

$$\begin{cases} L:\ 2 = 2x + y \\ M:\ 0 = x - 2 \\ T:\ 0 = -x - y \end{cases} \quad \Longrightarrow \quad \begin{cases} x = -y \\ x = 2 \\ y = -2. \end{cases}$$

Thus, the cross-section is:

$$\sigma_T = \frac{8}{3}\pi\frac{\alpha^2}{m_e^2}\left(\frac{\hbar}{c}\right)^2 = \frac{8\pi}{3}\frac{\alpha^2}{(m_e c^2)^2}(\hbar c)^2 \sim 64\ \mathrm{fm}^2 = 64 \cdot 10^{-2}\ \mathrm{b}. \tag{A.0.10}$$

Appendix B

Solution to BRST exercise

I present in this Appendix the solution for the exercise 20, from the section §8.3.1.

We can compute:

$$\delta_\theta(\mathcal{L}_{QCD}^{ghost}) = -\theta B^a \partial_\mu \mathcal{D}^{\mu ab} c^b + g_s \overline{c}^a \partial_\mu \theta f^{abc} \frac{1}{2} \partial^\mu (c^b c^c) - \\ - g_s f^{abc} \overline{c}^a \theta \partial_\mu \left[(\partial^\mu c^c) c^b + g_s f^{cde} A^{\mu e} c^d c^b - \frac{1}{2} g_s f^{bde} A^{\mu c} c^d c^e \right] \tag{B.0.1}$$

$$= -\theta B^a \partial_\mu \mathcal{D}^{\mu ab} c^b + \\ + g_s f^{abc} \overline{c}^a \theta \partial_\mu \left[\frac{1}{2} \partial^\mu (c^b c^c) - \partial^\mu (c^c) c^b \right] + \\ + g_s^2 \overline{c}^a \theta \partial_\mu \left(f^{abc} f^{cde} A^{\mu e} c^d c^b - \frac{1}{2} f^{abc} f^{bde} A^{\mu c} c^d c^e \right) \tag{B.0.2}$$

where we can write:

$$g_s f^{abc} \overline{c}^a \theta \partial_\mu \left[\frac{1}{2} \partial^\mu (c^b c^c) - \partial^\mu (c^c) c^b \right] = 0 \tag{B.0.3}$$

because the fields anticommute and f^{abc} are antisymmetric, and we can see:

$$f^{abc} f^{cde} A^{\mu e} c^d c^b - \frac{1}{2} f^{abc} f^{bde} A^{\mu c} c^d c^e \tag{B.0.4}$$

$$= \left(f^{aex} f^{xdc} - \frac{1}{2} f^{axc} f^{xde} \right) A^{\mu c} c^d c^e \tag{B.0.5}$$

$$= \frac{1}{2} \left(f^{aex} f^{xdc} + f^{adx} f^{xec} + f^{acx} f^{xde} \right) A^{\mu c} c^d c^e \tag{B.0.6}$$

$$= 0 \tag{B.0.7}$$

by Jacobi. ■

Appendix C

Project ideas

I collect in this chapter all the project ideas arised during the lectures.

From **Chapter 3 - ϕ^3 theory at one-loop**:

- Renormalization of ϕ^3 in $6d$ at 2-loops ($\overline{MS}$).
- Renormalization of ϕ^3 in $6d$ in MOM scheme (need to compute finite point of $I_3^{(1)[6-2\epsilon]}$).
- Study UV properties of ϕ^4 theory.

From **Chapter 4 - Loop integration methods in dimensional regularization**:

- Study different integral parametrizations, e.g. Baikov, Mellin Barnes etc.
- Study integration-by-parts reduction and differential equations for multi-loop integrals.
- Study graph theory connection to Symanzik polynomials.

From **Chapter 5 - Renormalization of QED at one-loop**:

- Renormalization of related theories e.g. scalar QED.
- Complete renormalization of QED using the on-shell scheme.
- Path integral methods to prove Ward-Takahashi identity.
- Proof of Furry's theorem.

From **Chapter 6 - Renormalization of QCD at one-loop**:

- Compute $\delta_1^{C(1)}$, $\delta_C^{(1)}$ and verify the relation: $\delta_1^{(1)} - \delta_{\psi_g}^{(1)} = \delta_1^{(1)} - \delta_C^{(1)}$.

- Explore the derivation of Ward-Takahashi and Slavnov-Taylor identities using path integral methods.

- Consider $\delta_{g_S}^{(1)}$ in super-symmetric extensions / relatives of QCD ($\mathcal{N} = 1$ / $\mathcal{N} = 4$ super Yang-Mills).

- Use the spinor-helicity method to derive 4-gluon scattering. Explore extensions to higher multiplicity with off-shell recursion relations.

From **Chapter 7 - The renormalization group**:

- Review Wilson's approach to renormalization.

- Describe the parameters of the Standard Model and their renormalization group equations.

- Explore how locality of quantum field theories and ensure quantum fluctuations a short distances may be renormalized.

From **Chapter 8 - Unitarity and gauge invariance**:

- Explore triple cut constraints for vertex function such as:

$$\gamma^* \longrightarrow e^+e^- \tag{C.0.1}$$

  in QED (massless limit interesting enough).

- Describe how the backround field method can be used to compute the β function via an effective action.

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