Notes from the course on:

Advanced Quantum Fields Theory

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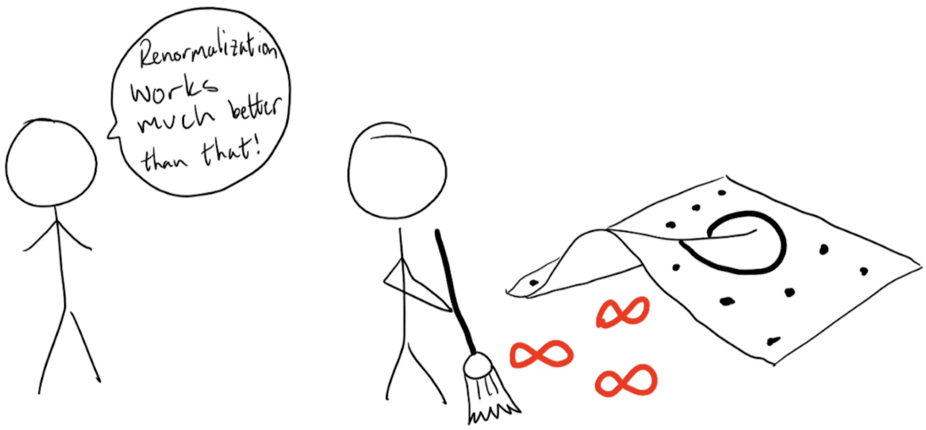
A.A. 2025-2026



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Legal Information

Let no one ignorant of geometry
enter my door.

Plato

Preface

In this document I aim to collect my notes based on the material from the course "**Advanced Quantum Fields Theory**", taught by Prof. S. D. Badger and attended at the *University of Turin* in the academic year 2025-2026. I also include references to various books (more or less useful depending on the desired level of depth). These notes are a rewritten version of the notes I took during the lectures, so the main source is the material presented by the professor. However, textbooks are essential for a full understanding of the topics. During the course, several books were recommended (listed in the Bibliography).

This course should be viewed as the third part of a three-part series on QFT; therefore, certain topics, notions, and concepts will be assumed as prior knowledge. For any review, please refer to the lecture notes on Introduction and Foundations of QFT. Also the notes of Phenomenology of foundamental interactions could be useful. You can see some of the exercises present in the notes following this link. M. Nebbia's note from the course *Complementi di Teoria di Campi*, taught by Prof.s L. Magnea and G. Passarino, are very useful. I posted here a list of the corresponding chapter.

These notes should clearly be understood as personal, neatly rewritten lecture notes. Any oversights, mistakes, or inaccuracies are due to my own limitations. Moreover, I wrote these notes mainly to "explain" the subject to myself, so some sections may appear overly detailed or, conversely, too superficial depending on the reader. In any case, I hope they may still be useful to someone. I also hope that I have managed to produce a clear and well-structured document.

Sometimes I may not explicitly reference a particular textbook or past course; in such cases, I am referring to my own notes on that topic. A collection of my notes is available on my personal GitHub page: gCembalo.github.io.

Any error or typo can be reported to my personal email:
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Contents

Chapter 1

Intoduction

In this course we want to develop the formalism of Quantum Field Theory to describe:

- *Loop computations*: quantum corrections in perturbation theory.
- The origin of *UV divergences* and the method of *renormalization.*
- *Renormalization scale indipendence* and the *renormalization group.*
- *Gauge invariance* and *unitarity* for non-abelian gauge theories.

The course structure is: a brief review of QFT, the $\lambda\phi^3$ theory, we will study the loop computation methods, we will see how to renormalize the QED and QCD to develop the formalism for the renormalization group, but the last part of the course is unitarity, "cuts" and non-abelian gauge invariance.

As presequisites we need to know how to derive Feynman rules and tree-level ampludes, we must be familiar with LSZ formalism and the correction between Green's functions and S-matrix elements and we must have basic knowledge of abelian, and non-abelian, gauge theory (QED and QCD).

Basic QFT history. In 1927 P. A. M. Dirac find a *quantum theory of electrons*, but with one big problem, the electron self-energy is infinity. Between 1947 and 1949 Feynman, Schwinger and Tomonaga develop the *renormalization of QED*. We can think that in that moment they build the QFT.

Note that was also contributions from Kramers, Bethe and Dyson.

Later on the physicist develop the Yang-Mills theory, the QCD, Weinberg (and other in '67) develop the Standard Model and the EW unification, 'T Hooft and Veltman in '72 contributed to EW renormalization.

Between 1972 and 1975 has been developed, by Wilson based on preceding work by Stwekelber, Peterman, Gell-Mann, Low and other, the renormalization group theory.

Chapter 2

Loop correction in scalar Quantum Field Theory

In this introductory chapter we review the formalism for perturbative scalar field theory. You can see my lecture notes from the other two QFT course.

Note. In this chapter we will represent the Feynman diagrams for scalar fields with solid lines, and not with dashed.

2.1 Basic formalism

The lagrangian density for a real scalar field, in $4d$ Minkowski spacetime using diagonal metric:

$$g^{\mu\nu} = \text{diag}(1, -1, -1, -1) \tag{2.1.1}$$

with an arbitrary potential $V(\phi)$ is:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi) \tag{2.1.2}$$

where $\phi = \phi(x^\mu)$ con x^μ is a 4-vector in Minkowski spacetime. We will take the convention $\hbar = c = 1$ (you can read the Appendix A) throughout. The classical equation of motion (Euler-Lagrange equations) are:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \Longrightarrow \quad \partial_\mu \partial^\mu \phi + \frac{\partial V}{\partial \phi} = 0. \tag{2.1.3}$$

As we learn, we can quantize the theory by imposing *canonical commutation relations* between equal time fields and canonical momenta $\pi(x^\mu)$:

$$\pi(x^\mu) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \tag{2.1.4}$$

that we choose:

$$\left[\phi(t,\vec{x}),\phi(t,\vec{x}')\right]=0 \tag{2.1.5}$$
$$\left[\pi(t,\vec{x}),\pi(t,\vec{x}')\right]=0 \tag{2.1.6}$$
$$\left[\pi(t,\vec{x}),\phi(t,\vec{x}')\right]=-i\delta^{(3)}(\vec{x}-\vec{x}'). \tag{2.1.7}$$

The hamiltonian is then:

$$H=\int \mathrm{d}^3x\left(\frac{\pi^2}{2}+\frac{(\vec{\partial}\phi)^2}{2}+V(\phi)\right) \tag{2.1.8}$$

with:

$$(\vec{\partial}\phi)^2=(\partial_i\phi)(\partial^i\phi)=\sum_{i=1}^{3}(\partial_i\phi)^2. \tag{2.1.9}$$

Doing so, the field ϕ is now interpreted as an operator on Hilbert space:

$$\phi \quad\longrightarrow\quad \hat{\phi} \tag{2.1.10}$$

satisfying the Heisemberg equation of motion:

$$i\frac{\partial\hat{\phi}}{\partial t}=\left[\hat{\phi},H\right]. \tag{2.1.11}$$

Remember, we assume that, not only exist a Poincaré invariant vacuum state, but also that the solution of the theory may be obtain ($\hat{\phi}$ and the Fock space) using the n-point correlation functions between fields:

$$G_n(x_1,\dots,x_n)=\langle 0|\,T\left[\phi(x_1),\dots,\phi(x_n)\right]|0\rangle \tag{2.1.12}$$

where T represents time ordering. For the two point function the equations of motion imply:

$$\frac{\partial}{\partial x_1^{\mu}}\frac{\partial}{\partial x_{1\mu}}G_2(x_1,x_2)+\langle 0|\,T\left[\left(\frac{\partial V}{\partial\phi}\right)(x_1)\,\phi(x_2)\right]|0\rangle=-i\delta^{(4)}(x_2-x_1) \tag{2.1.13}$$

$$\Box_{x_1}G_2(x_1,x_2)+\langle 0|\,T\left[V'(x_1)\phi(x_2)\right]|0\rangle=-i\delta^{(4)}(x_2-x_1) \tag{2.1.14}$$

or for the general case:

$$\Box_{x_1}G_n(x_1,\dots,x_n)+\langle 0|\,T\left[V'(x_1)\dots\phi(x_n)\right]|0\rangle= \\ =-i\delta^{(4)}(x_2-x_1)G_{n-1}(x_2,\dots,x_n). \tag{2.1.15}$$

2.1.1 The generating functional for correlation functions

Let us remind ourselves of the solution for G_2 in the free theory:

$$G_2(x_1, x_2) = \langle 0| T[\phi(x_1)\phi(x_2)] |0\rangle \tag{2.1.16}$$
$$= D_F(x_1 - x_2) \tag{2.1.17}$$
$$= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{ik(x_2-x_1)}}{k^2 - m^2 + i\emptyset^+} \tag{2.1.18}$$

where $D_F(x_1 - x_2)$ indicates this is the *Feynman propagator* for a particle of mass m (using $V(\phi) = \frac{1}{2}m^2\phi^2$) and we have introduced a small positive immaginary term $\emptyset^+$ in the denominator to resolve the poles on the real axis.

For the interacting theory we may write (thanks its semplicity) the solution using the *path integral* formalism:

$$\langle 0| T[\phi(x_1)\phi(x_2)] |0\rangle = \lim_{t\to\infty} \frac{\int \mathcal{D}_\phi\, \phi(x_1)\phi(x_2)\, \exp\left\{i \int_{-t}^{t} \mathrm{d}^4x\, \mathcal{L}\right\}}{\int \mathcal{D}_\phi\, \exp\left\{i \int_{-t}^{t} \mathrm{d}^4x\, \mathcal{L}\right\}} \tag{2.1.19}$$

which we can solve using a generating functional $Z[J]$ with a source term J:

$$Z[J] = \int \mathcal{D}_\phi\, \exp\left\{ i \int \mathrm{d}^4x\, (\mathcal{L} + J(x)\phi(x)) \right\} \tag{2.1.20}$$

so the correlation function, $G_2(x_1, x_2)$, is obtained via functional derivatives of $Z[J]$ with respect to the source term J:

$$G_2(x_1, x_2) = \langle 0| T[\phi(x_1)\phi(x_2)] |0\rangle \tag{2.1.21}$$
$$= \frac{1}{Z[0]} \left(-i\frac{\delta}{\delta J(x_1)}\right) \left(-i\frac{\delta}{\delta J(x_2)}\right) Z[J]\Bigg|_{J=0}. \tag{2.1.22}$$

2.2 Two point correlation function for ϕ^3 theory in position and momentum space

We will only consider perturbative solutions here. Let us now specify a particular interaction, the ϕ^3 theory, in the coordinates and momentum space. We take:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \tag{2.2.1}$$
$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{2.2.2}$$
$$\mathcal{L}_I = -\frac{\lambda}{3!}\phi^3. \tag{2.2.3}$$

We may expand $Z[J]$ as follows:

$$\begin{aligned} Z[J] &= \int \mathcal{D}_\phi \exp\left\{ i \int \mathrm{d}^4x \, (\mathcal{L}_0 + \mathcal{L}_I) + J\phi \right\} && (2.2.4)\\ &= \int \mathcal{D}_\phi \exp\left\{ i \int \mathrm{d}^4x \, \mathcal{L}_I \right\} \exp\left\{ i \int \mathrm{d}^4x \, \mathcal{L}_0 + J\phi \right\} && (2.2.5)\\ &= \int \mathcal{D}_\phi \exp\{ i S_I[\phi] \} \exp\left\{ i \int \mathrm{d}^4x \, \mathcal{L}_0 + J\phi \right\} && (2.2.6)\\ &= \int \mathcal{D}_\phi \sum_{n=0}^{\infty} \frac{1}{n!} \left(i S_I[\phi] \right) \exp\left\{ i \int \mathrm{d}^4x \, \mathcal{L}_0 + J\phi \right\}. && (2.2.7) \end{aligned}$$

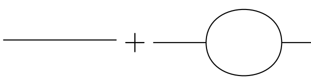
For the free theory one can derive the 2-point correlation function from the relation:

$$\begin{aligned} Z_0[J] = Z_0[0] \exp&\left\{ -\frac{1}{2} \int \mathrm{d}^4x \, \mathrm{d}^4y \, J(x) D_F(x-y) J(y) \right\} && (2.2.8)\\ \implies \quad & G_{2,0}(x_1, x_2) = D_F(x_1 - x_2) && (2.2.9) \end{aligned}$$

so, to solve the interacting theory we can expand around the free propagator:

$$\frac{1}{Z_0[0]} \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \Bigg(1 + \underbrace{\frac{i\lambda}{6} \int \mathrm{d}^4 z \left(\frac{\delta}{\delta J(z_1)} \right)^3}_{=0} + \\ + \left(\frac{i\lambda}{6} \right)^2 \int \mathrm{d}^4 z_1 \mathrm{d}^4 z_2 \left(\frac{\delta}{\delta J(z_1)} \right)^3 \left(\frac{\delta}{\delta J(z_2)} \right)^3 + \ldots \Bigg) Z_0[J] \Bigg|_{J=0} \qquad (2.2.10)$$

$$\begin{aligned} &= D_F(x_1 - x_2) - \\ &\quad - \frac{\lambda^2}{2} \int \mathrm{d}^4 z_1 \mathrm{d}^4 z_2 \, D_F(x_1 - z_1) D_F(z_1 - z_2)^2 D_F(z_2 - x_2) + \\ &\quad + \mathcal{O}(\lambda^4) && (2.2.11) \end{aligned}$$


$+ \mathcal{O}(\lambda^4)$

where we didn't draw the disconnected diagrams.

We already have an issue with this expression since the point $z_1 = z_2$ causes a singularity. The divergence is simpler to see in momentum space, so we take the Fourier transform of $G_2(x_1, x_2)$ (recalling the espression for $D_F(x_1 - x_2)$ (2.1.18)):

$$\begin{aligned} &D_F(x_1 - z_1) D_F(z_1 - z_2)^2 D_F(z_2 - x_2) = \\ &\quad = \int \left(\prod \frac{\mathrm{d}^4 k_i}{(2\pi)^4} \right) \frac{e^{ik_1(z_1 - x_1)} e^{ik_2(z_2 - x_1)} e^{ik_3(z_2 - z_1)} e^{ik_4(z_2 - x_2)}}{\prod_{i=1}^4 (k_i^2 - m^2 + i\emptyset^+)} && (2.2.12) \end{aligned}$$

where we can rewrite the argument of the exponential to collect on z_1 and z_2:

$$\exp\{iz_1(k_1 - k_2 - k_3) - iz_2(k_4 - k_3 - k_2) - ik_1x_1 + ik_4x_2\} \tag{2.2.13}$$

in this way we can compute $\tilde{G}_2$ by evaluating the integral in z_1 and z_2 in (2.2.11). In particular $\int \mathrm{d}^4z_1$ gives us $(2\pi)^4\delta^{(4)}(k_1 - k_2 - k_3)$, that with $\int \mathrm{d}^4k_3/(2\pi)^4$ sets $k_3 = k_1 - k_2$, then $\int \mathrm{d}^4z_2$ gives $(2\pi)^4\delta^{(4)}(k_4 - k_1)$ and then $\int \mathrm{d}^4k_4/(2\pi)^4$ sets $k_4 = k_1$. Putting everithing together:

$$\tilde{G}_2(p^2, m^2) = \int \mathrm{d}^4\tilde{x}\, e^{ip\tilde{x}}\, G_2(x_1, x_2) \tag{2.2.14}$$

with $\tilde{x} = x_1 - x_2$ and:

$$\begin{aligned} G_2(x_1, x_2) &= D_F(x_1 - x_2) - \\ &- \frac{\lambda^2}{2} \int \frac{\mathrm{d}^4k_1}{(2\pi)^4} \frac{\mathrm{d}^4k_2}{(2\pi)^4} \frac{e^{-ik_1(x_1 - x_2)}}{D(k_1, m)^2\, D(k_2, m)\, D(k_1 - k_2, m)} \end{aligned} \tag{2.2.15}$$

where we introduce:

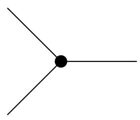
$$D(k, m) = k^2 - m^2 + i\emptyset^+. \tag{2.2.16}$$

Now, the integral over $\tilde{x} = x_1 - x_2$ can be performed giving a $\delta^{(4)}(p - k_1)$ factor, followed by the integral in d^4k_1 fixing $k_1 = p$, so gives us:

$$\begin{aligned} \tilde{G}_2(p^2, m^2) &= \frac{i}{D(p, m)} + \\ &+ \frac{(-i\lambda)^2}{2} \left(\frac{i}{D(p, m)}\right)^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{i}{D(k, m)} \frac{i}{D(k - p, m)} + \mathcal{O}(\lambda^4). \end{aligned} \tag{2.2.17}$$

We like this result, but we know, from other courses, that Feynman introduced a diagrammatic way to arrive at the same expression avoiding the lenght algebra. For the $\lambda\phi^3$ theory we have the Feynman rules:

- Interaction vertex:

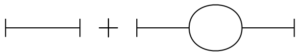

$= -i\lambda$

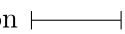
- Internal lines:

——— $= \dfrac{i}{D(k, m)}$

- Divide by the symmetry factor of the diagram.

In this way we have:

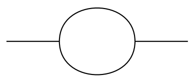
$$\tilde{G}_2(p^2, m^2) =$$  $$+\ \mathcal{O}(\lambda^4)$$

where we have used the notation  to indicate this is a correlations function and so *propagators are included* on external lines, unlike the case of S-matrix elements where diagrams are amputated.

Note that we must include the symmetry factor of 1/2 (to the loop) when applying the Feynman rules.

2.2.1 Identifying the problem

The one-loop correction to the propagator, usually reffered to ad *one-loop self energy*, is:

$$\Sigma^{(1)}(p^2, m^2) \ = -i$$ 

where:

$$\Sigma^{(1)}(p^2, m^2) = \frac{i\lambda}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{D(k, m) D(k-p, m)} \tag{2.2.18}$$

so that:

$$\tilde{G}_2(p^2, m^2) = \frac{i}{D(p, m)} \left(1 + \frac{\Sigma^{(1)}(p^2, m^2)}{D(p, m)} + \mathcal{O}(\lambda^4) \right). \tag{2.2.19}$$

Wrote this we can explicit the limit $|k| \to \infty$:

$$\Sigma^{(1)}(p^2, m^2) \quad \longrightarrow \quad \frac{i\lambda}{2} \int_0^\infty \frac{\mathrm{d}|k| \, |k|^3 \, \mathrm{d}^3\Omega}{|k|^4} \propto \int_0^\infty \frac{\mathrm{d}|k|}{|k|} \tag{2.2.20}$$

hence the self-energy has a logaritmic divergence in the UV, and we must include a cut-off to compute the integral:

$$\int_\delta^\Lambda \frac{\mathrm{d}|k|}{|k|} = \log\left(\frac{\Lambda}{\delta}\right). \tag{2.2.21}$$

Understanding how to treat such UV divergences through *rinormalizzation*[1], will be the main topic of this course.

[1] Note that including a cut-off is only one way (of many) to dominate the infinity.

Chapter 3

$\lambda\phi^3$ theory at 1 loop

We will consider our study of ϕ^3 theory. This time we specify a generale space-time dimension, d, so we have:

$$S = \int \mathrm{d}^4x\, \mathcal{L} \tag{3.0.1}$$

with the same lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{3!}\phi^3. \tag{3.0.2}$$

The metric is taken to be *mostly minus* hence:

$$g^{\mu\nu} = \mathrm{diag}\{1, -1, \ldots, -1\}. \tag{3.0.3}$$

Before finding a resolution to the problem of UV divergences our first task is to classify the possible divergent graph.

In particular we will saw how to renormalize UV divergences in ϕ^3 in $4d$, with a mass shift (and redefining $\langle\phi\rangle$), we will introduce methods for loop integrals, like dimensional regularization and Feynman parameters. We will saw renormalized perturbation theory depends on an arbitrary scale μ_R at fixed order.

Note. In this chapter, as the previous one, we will represent the Feynman diagrams for scalar fields with solid lines, and not with dashed.

3.1 Superficial degree of divengence

We can repeat the argument from chapter §2 in d-dimension concerning the UV divergence of the one loop self-energy:

$$\Sigma^{(1)[d]}(p^2, m^2) = -i\frac{\lambda^2}{2}I_2^{(1)[d]} \tag{3.1.1}$$

with the integral:

$$I_2^{(1)[d]} = \int_k \frac{1}{D(k,m)D(k-p,m)} \tag{3.1.2}$$

and where we write:

$$\int_k = \int \frac{\mathrm{d}^d k}{(2\pi)^4}.$$

In the limit $|k| \to \infty$ we have:

$$I_2^{(1)[d]} \underset{|k|\to\infty}{\longrightarrow} \int \frac{\mathrm{d}|k|\,|k|^{d-1}\mathrm{d}^{d-1}\Omega}{|k|^4(2\pi)^4} \tag{3.1.3}$$

$$= \underbrace{\int \mathrm{d}|k|\,|k|^{d-5}} \int \frac{\mathrm{d}^{d-1}\Omega}{(2\pi)^d} \tag{3.1.4}$$

therefore the integral underlined, for any dimension $d \geq 4$ diverges, and its divergence in $d = 4$ is logarithmic.

There is a natural lower cut-off for the energy, the mass m, so using a cut-off regularisation scale Λ:

$$\Sigma^{(1)[4]}(p^2, m^2) \underset{|k|\to\infty}{\longrightarrow} c \log\left(\frac{\Lambda^2}{m^2}\right) \tag{3.1.5}$$

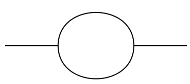
with c a constant and we use Λ as a UV regulator.

NB. Since there is non-zero mass there is a natural lower band for the $|k|$ integral.

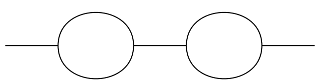
We can perform this analysis for a general (one-particle-irreducible, 1PI) graph $\Gamma_E(n, L)$ where:

- E the number of external lines
- n the number of internal lines
- L the number of loops.

The graph:



is 1PI, while the graph:



is not.

The UV power counting gives us the *superficial degree of divergence*, $\omega(\Gamma_E)$:

$$\omega(\Gamma_E) = dL - 2n. \tag{3.1.6}$$

The extension to higer loop assumes scaling all internal momenta to ∞ at the same rate. We may try to conclude:

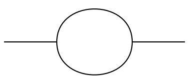
- $\omega(\Gamma_E) < 0$ means *convergence.*
- $\omega(\Gamma_E) = 0$ means *logarithmic divergence.*
- $\omega(\Gamma_E) > 0$ means *power low divergence.*

This statement can be made concrete by considering also the superficial degree of divergence of each subgraph:

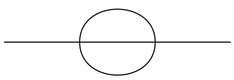
Weinberg's Theorem: A graph Γ is convergent in the UV if $\omega(\Gamma) < 0$ and $\omega(\Gamma_{sub}) < 0$ for every subgraph Γ_{sub} of Γ.

NB. The superficial degree of divergence does not account of potential cancellations between graphs with the same number of external legs and loop order, so contributing to the same correlation function G_n.

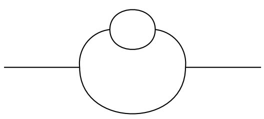
Examples of divergences:



$$\begin{cases} n = 2 \\ E = 2 \\ L = 1 \end{cases} \quad ; \quad \omega(\Gamma_2) = d - 4$$



$$\begin{cases} n = 3 \\ E = 2 \\ L = 2 \end{cases} \quad ; \quad \omega(\Gamma_2) = 2d - 6 = 2(d-3)$$



$$\begin{cases} n = 5 \\ E = 2 \\ L = 2 \end{cases} \quad ; \quad \omega(\Gamma_2) = 2d - 10 = 2(d-5)$$

In this last case we might be tempted to say convergent in $d = 4$, but subgraph diverges in $d = 4$ (the one-loop in the upper line).

We may express ω in terms of the dimensions of the fields and couplings. For a good general scalar theory with p-point interaction, so:

$$\mathcal{L}_{int} = -\frac{\lambda}{p!}\phi^p \tag{3.1.7}$$

we find that:

$$L = n - V + 1 \qquad , \qquad n = \frac{pV - E}{2} \tag{3.1.8}$$

so:

$$\begin{aligned}
\omega(\Gamma) &= dL - 2n && (3.1.9)\\
&= d(n - V + 1) - 2n && (3.1.10)\\
&= (d-1)\frac{(pV - E)}{2} - dV + d && (3.1.11)\\
&= d - E\left(\frac{d-2}{2}\right) - V\left(d - p\left(\frac{d-2}{2}\right)\right) && (3.1.12)\\
&= d - E\frac{[\phi]}{[m]} - V\frac{[\lambda]}{[m]} && (3.1.13)
\end{aligned}$$

where we have used, from the lagrangian:

$$\begin{aligned}
[m^2\phi^2] &= 2[m] + 2[\phi] = d[m] && (3.1.14)\\
\implies \qquad [\phi] &= \frac{(d-2)}{2}[m] && (3.1.15)\\
[\lambda\phi^p] &= d[m] && (3.1.16)\\
\implies \qquad [\lambda] + p[\phi] &= d[m] && (3.1.17)\\
\frac{[\lambda]}{[m]} &= d - \left(\frac{d-2}{2}\right)p. && (3.1.18)
\end{aligned}$$

Our ability to renormalize will rely on the number of UV divergences that we may encounter in the correlation functions. We would like a finite number of divergent (sub)-graph. In terms of the dimension of λ we classify cases as:

- **Super-renormalizable**: if $[\lambda]/[m] > 0$ and a graph Γ has $\omega(\Gamma) < 0$, at loop order L, then the additional loop correction to Γ (some values for $d, E, [\phi]/[m]$, higher L meand higher V) will also be convergent (will decrease $\omega(\Gamma)$), so this means a finite number of divergent graphs.
- **Renormalizable**: if $[\lambda]/[m] = 0$ then we have ∞ number of divergent graphs, but only for limited number of extenal lines. However, the number of divergent sub-graphs is finite.
- **Non-renormalizable**: if $[\lambda]/[m] < 0$. Every 1PI graph is divergent if we go to a sufficiently high loop order.

3.2 Computing $\Sigma^{(1)}$ in dimensional regularization

Having estabilshed that the self-energy is divergent we may calculate only after introducing a *regularization* method.

We already demonstrated a simple *cut-off* would render the integral finite, but breaks Poincaré invariance, so is not predered. We can perform *dimensional regularization* where d is analytically continued to be non-integer:

$$d = 4 - 2\epsilon \qquad , \qquad \epsilon \ll 0 \tag{3.2.1}$$

note that instead 4 we can use d_0, the dimension for $\mathcal{L}$. Dimensional regularization is an analytic continuation to a continuous dimension, and it preserves many symmetries (Poincaré, abelian and non-abelian gauge) and so it can be generalizes well to QED and QCD. For this reason we may as well introduce it immediately. Later on in the chapter we will list other possibilities.

We can now simply compute:

$$\Sigma^{(1)[d]} = -i\frac{\lambda^2}{2} I_2^{(1)[d]} \tag{3.2.2}$$

where:

$$I_2^{(1)[d]} = \int_k \frac{1}{D(k,m)D(k-p,m)}. \tag{3.2.3}$$

3.2.1 Feynman parameters

Feynman, and similarly Schwinger, notice that it was convenient to observe (you can try it in Mathematica) the relation:

$$\frac{1}{AB} = \int_0^1 \mathrm{d}\alpha_1 \int_0^1 \mathrm{d}\alpha_2 \, \frac{\delta(1-\alpha_1-\alpha_2)}{(\alpha_1 A + \alpha_2 B)^2}. \tag{3.2.4}$$

We will study the generalization to more denominators with arbitrary powers later, for now consider just 2 denominator. We write, computing one delta, because it's convenient:

$$I_2^{(1)[d]} = \int_k \int_0^1 \mathrm{d}\alpha \, \frac{1}{(\alpha D(k-p,m) + (1-\alpha)D(k,m))^2} \tag{3.2.5}$$

completing the square in the denominatos yields:

$$\alpha((k-p)^2 - m^2 + i\emptyset^+) + (1-\alpha)(k^2 - m^2 + i\emptyset^+) = \tag{3.2.6}$$

$$= k^2 - m^2 + i\emptyset^+ - 2\alpha k \cdot p + \alpha p^2 \tag{3.2.7}$$

$$= (k - \alpha p)^2 - \alpha p^2 + \alpha^2 p^2 - m^2 + i\emptyset^+ \tag{3.2.8}$$

$$= (k')^2 - \Delta \tag{3.2.9}$$

where we use:

$$\Delta = -\alpha(1-\alpha)p^2 + m^2 - i\emptyset^+ \tag{3.2.10}$$

$$k' = k - \alpha p. \tag{3.2.11}$$

We can now re-order integration in α and k (changing variable from k' to k) to write:

$$I_2^{(1)[d]} = \int_0^1 \mathrm{d}\alpha \int_k \frac{1}{(k^2-\Delta)^2}. \tag{3.2.12}$$

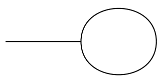
Note. If the numerator has dependence on k we would pick up α dependence in the numerator due to the shift $k' = k - \alpha p$.

3.2.2 The tadpole integral

In $I_2^{(1)[d]}$ (3.2.12) we have a standard integral in itself that we can solve; for this reason we can consider a general **tadpole integral**:

$$T_n^{[d]}(\Delta) = \int_k \frac{1}{(k^2-\Delta)^n}. \tag{3.2.13}$$

A tadpole graph would look like:



The first step is to Wick rotate into Euclidean kinematics:

$$k^0 = ik_E^0 \qquad , \qquad \vec{k} = \vec{k}_E \tag{3.2.14}$$

such that:

$$k^2 = (k^0)^2 - |\vec{k}|^2 \tag{3.2.15}$$
$$= -(k_E^0)^2 - |\vec{k}_E|^2 \tag{3.2.16}$$
$$= -k_E^2 \tag{3.2.17}$$

and:

$$\mathrm{d}^d k = i\mathrm{d}^d k_E. \tag{3.2.18}$$

The tranformation (remember that the Wick rotation rotate the axis in anti-clock sense) does not cross the poles from the denominator[1], see fig. 3.1, and note that they are:

$$k^2 - \Delta = (k^0)^2 - |\vec{k}|^2 - \Delta = 0 \tag{3.2.21}$$

$$\Longrightarrow \qquad k^0 = \pm\sqrt{|\vec{k}|^2+\Delta} = \pm\sqrt{|\vec{k}| + \mathrm{Re}\{\Delta\}} \mp i\emptyset^+ = k_\pm^0. \tag{3.2.22}$$

[1]Note that this is more than a replacement. See Peskin [2] and Sterman [3]. When we perform the Wick rotation we must do it in anti-clock sense, because otherwise we'll cross the pole from the denominator, see fig. 3.1. The rotation is:

$$\int_{-\infty}^{+\infty} \mathrm{d}k^0 = \oint_{C_R} \mathrm{d}k^0 \qquad , \qquad \text{with } k^0 \in \mathbb{C} \tag{3.2.19}$$

and using the Cauchy's theorem:

$$= \oint_{C_I} \mathrm{d}k^0 = \int_{-i\infty}^{+i\infty} \mathrm{d}k_I^0 = i\int_{-\infty}^{+\infty} \mathrm{d}k_E^0 \tag{3.2.20}$$

where we have k_I^0 immaginary, but k_E^0 is real.

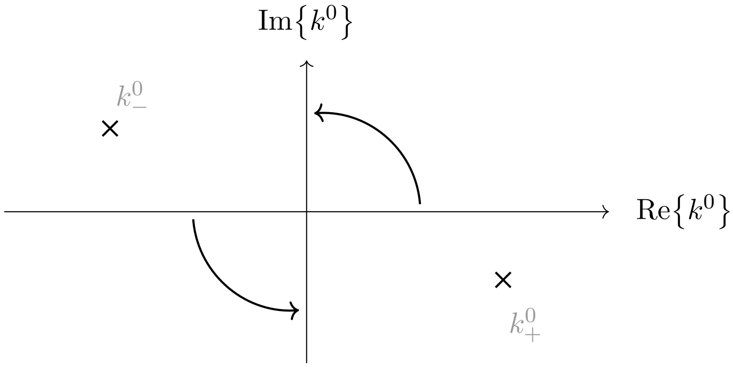


Figure 3.1

Therefore:

$$T_n^{[d]}(\Delta) = i(-1)^n \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \frac{1}{(k_E^2+\Delta)^n} \tag{3.2.23}$$

$$= i(-1)^n \int \frac{|k_E|^{d-1}\,\mathrm{d}|k_E|}{(|k_E|^2+\Delta)^n} \underbrace{\int \frac{\mathrm{d}^{d-1}\Omega}{(2\pi)^d}}_{=2\pi^{d/2}/\Gamma(d/2)} \tag{3.2.24}$$

where we used the Gamma function:

$$\Gamma(x) = \int_0^x \mathrm{d}z\, z^{x-1} e^{-z}.$$

The integration in $|k_E|$ is also straightforward, because we can call $|k_E| = x$, then do the sostitution $x^2 = t$ and then $y = \Delta/(t+\Delta)$ to get:

$$\int_0^\infty \mathrm{d}x \frac{x^{d-1}}{(x^2+\Delta)^n} \underset{x^2=t}{=} \int_0^\infty \frac{\mathrm{d}t}{2} \frac{t^{d/2-1}}{2(t+\Delta)^n} \tag{3.2.25}$$

$$= \frac{1}{2}\int_0^1 \mathrm{d}y\, \Delta^{d/2-n} y^{n-1-d/2}(1-y)^{d/2-1} \tag{3.2.26}$$

we can reconize the Euler beta function:

$$B(q,b) = \int_0^1 \mathrm{d}x\, x^{a-1}(1-x)^{b-1}$$
$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

in this way we have:

$$\int \frac{|k_E|^{d-1}\,\mathrm{d}|k_E|}{(|k_E|^2+\Delta)^n} = \frac{1}{2}\Delta^{d/2-n}\frac{\Gamma\left(n-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma(n)} \tag{3.2.27}$$

so we have a result for the tadpole:

$$T_n[d](\Delta) = \frac{i(-1)^n}{(4\pi)^{d/2}}\Delta^{d/2-n}\frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)}. \tag{3.2.28}$$

3.2.3 The massless self-energy

Before completing $I_2^{(1)[d]}(p^2, m^2)$ we can try the simpler case, at one loop in d dimension, with $m = 0$. We have to use the result (3.2.28) in d dimension with $n = 2$, so:

$$T_2^{[d]}(\underbrace{-\alpha(1-\alpha)p^2}_{\Delta \quad \text{dropping } i\emptyset^+}) = \frac{i}{(4\pi)^{d/2}}\left(-\alpha(1-\alpha)p^2\right)^{d/2-2}\frac{\Gamma(2-d/2)}{\Gamma(2)} \tag{3.2.29}$$

so we have:

$$I_2^{(1)[d]}(p^2, 0) = \int_0^1 \mathrm{d}\alpha\, T_2^{[d]}(\Delta) \tag{3.2.30}$$

$$= \frac{i}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}\alpha \frac{\Gamma(2-d/2)}{\Gamma(2)}\left(-\alpha(1-\alpha)p^2\right)^{d/2-2} \tag{3.2.31}$$

$$= \frac{i}{(4\pi)^{d/2}}(-p^2)^{d/2-2}\Gamma\left(2-\frac{d}{2}\right) B\left(\frac{d}{2}-1, \frac{d}{2}-1\right) \tag{3.2.32}$$

$$= \frac{i}{(4\pi)^{d/2}}(-p^2)^{d/2-2}\frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma^2\left(\frac{d}{2}-1\right)}{\Gamma(d-2)} \tag{3.2.33}$$

in $d = 4-2\epsilon$ dimensions, we have:

$$\frac{d}{2}-n \longrightarrow \frac{4-2\epsilon}{2}-2 = -\epsilon$$

so, computing (3.2.33) for $d = 4-2\epsilon$ we have:

$$I_2^{(1)[4-2\epsilon]}(p^2, 0) = \frac{i}{(4\pi)^{2-\epsilon}}(-p^2)^{-\epsilon}\frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} \tag{3.2.34}$$

where the UV pole can be made explicit through use of the property $\Gamma(1+z) = z\Gamma(z)$:

$$\Gamma(\epsilon) = \frac{\Gamma(1+\epsilon)}{\epsilon} \quad , \quad \Gamma(2-2\epsilon) = (1-2\epsilon)\Gamma(1-2\epsilon) \tag{3.2.35}$$

so we can conclude:

$$I_2^{(1)[4-2\epsilon]}(p^2, 0) = \frac{iC_\Gamma}{(4\pi)^2}\frac{1}{(1-2\epsilon)\epsilon}(-p^2)^{-\epsilon} \tag{3.2.36}$$

$$\text{with the coefficient:} \qquad C_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{(4\pi)^{-\epsilon}\Gamma(1-2\epsilon)}. \tag{3.2.37}$$

We may also expand in ϵ to observe the analytic structure:

$$I_2^{(1)[4-2\epsilon]}(p^2, 0) = \frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon}+2-\gamma_E+\log(4\pi)-\log\left(-p^2\right)+\mathcal{O}(\epsilon)\right) \tag{3.2.38}$$

where we expand $(-p^2)^{-\epsilon}$, and $(4\pi)^{\epsilon}$, with the log:

$$\left(\frac{-p^2}{4\pi}\right)^{-\epsilon} = \exp\left\{-\epsilon\log\left(\frac{-p^2}{4\pi}\right)\right\} \sim 1-\epsilon\Big[\log(-p^2)-\log(4\pi)\Big] \quad (3.2.39)$$

and we have the Euler-Mascheroni's constant:

$$\gamma_E = 0.57721 \quad (3.2.40)$$

that comes from the expansion of Gamma function:

$$\Gamma(\epsilon) \sim \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)$$

in particular we have to use:

$$\begin{aligned}
\Gamma(1+\epsilon) &= \epsilon\Gamma(\epsilon) \sim \epsilon\left(\frac{1}{\epsilon}-\gamma_E\right) = 1-\epsilon\gamma_E \\
\Gamma(1-\epsilon) &= -\epsilon\Gamma(-\epsilon) \sim -\epsilon\left(-\frac{1}{\epsilon}-\gamma_E\right) = 1+\epsilon\gamma_E \\
&\implies \quad \Gamma^2(1-\epsilon) \sim (1+\epsilon\gamma_E)^2 \sim 1+2\epsilon\gamma_E \\
\Gamma(1-2\epsilon) &= (-2\epsilon)\Gamma(-2\epsilon) \sim (-2\epsilon)\left(-\frac{1}{2\epsilon}-\gamma_E\right) = 1+2\epsilon\gamma_E.
\end{aligned}$$

3.2.4 The self-energy with mass dependence

Now focus on the same problem, with $n=2$ in d dimension, but with $m \neq 0$; we have:

$$I_2^{(2)[d]}(p^2,m^2) = \int_0^1 \mathrm{d}\alpha\, T_2^{[d]}\Big(\underbrace{-\alpha(1-\alpha)p^2+m^2}_{\hat{\Delta}\text{ droping } i\emptyset^+}\Big) \quad (3.2.41)$$

$$= \frac{i}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\int_0^1 \mathrm{d}\alpha\,\left(-\alpha(1-\alpha)p^2+m^2\right)^{d/2-2} \quad (3.2.42)$$

putting $d = 4-2\epsilon$:

$$I_2^{(2)[4-2\epsilon]}(p^2,m^2) = \frac{i}{(4\pi)^{2\epsilon}}\Gamma(\epsilon)(m^2)^{-\epsilon}\int_0^1 \mathrm{d}\alpha\left(\underbrace{1-\alpha(1-\alpha)\frac{p^2}{m^2}}_{\hat{\Delta}}\right)^{-\epsilon} \quad (3.2.43)$$

the UV divergence is the same:

$$\Gamma(\epsilon) = \frac{1}{\epsilon}\Gamma(1+\epsilon) \quad (3.2.44)$$

and the integral in α is finite so we can expand it in ϵ, and then integrate in α:

$$\int_0^1 \mathrm{d}\alpha\, \hat{\Delta}^{-\epsilon} = \int_0^1 \mathrm{d}\alpha \left(1 - \epsilon \log(\hat{\Delta}) + \dots\right) \tag{3.2.45}$$

and:

$$I_2^{(1)[4-2\epsilon]}(p^2, m^2) = \frac{i}{(4\pi)^{2-\epsilon}} \Gamma(1+\epsilon)(m^2)^{-\epsilon} \left(\frac{1}{\epsilon} - \int_0^1 \mathrm{d}\alpha\, \log(\hat{\Delta}) + \mathcal{O}(\epsilon)\right) \tag{3.2.46}$$

deriving a closed form for the integral over α is a straightforward exercise (you can see the linked file).

Exercise 1.

1. From the definitions $\beta_\pm = \frac{1}{2}(1 \pm \beta)$ and $\beta = \sqrt{1 - \frac{4m^2}{p^2}}$ show that $\beta_+\beta_- = m^2/p^2$.
2. Use the transformation $\alpha = y + \beta_+$ to show:
$$\hat{\Delta} = \frac{y(y+\beta)}{\beta_+\beta_-}.$$
3. Complete the integration over y to show:
$$\int_0^1 \mathrm{d}\alpha \log\left(\hat{\Delta}\right) = -2 + \beta \log\left(-\frac{\beta_+}{\beta_-}\right).$$

The final result is therefore:

$$I_2^{(1)[4-2\epsilon]}(p^2, m^2) = \frac{i\Gamma(1+\epsilon)m^{-2\epsilon}}{(4\pi)^{2-\epsilon}} \left(\frac{1}{\epsilon} + 2 - \beta \log\left(-\frac{\beta_+}{\beta_-}\right)\right) + \mathcal{O}(\epsilon) \tag{3.2.47}$$

$$= \frac{i}{(4\pi)^2} \left(\frac{1}{\epsilon} + 2 + \log(4\pi) - \gamma_E - \log(m^2) - \beta \log\left(-\frac{\beta_+}{\beta_-}\right)\right) + \mathcal{O}(\epsilon). \tag{3.2.48}$$

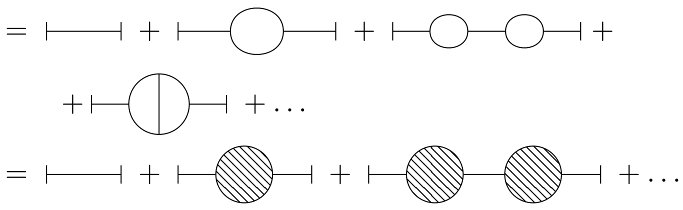
Putting this into the definition of $\Sigma^{(1)}$ (3.2.2) we find (at the first order):

$$\Sigma^{(1)[4-2\epsilon]}(p^2, m^2) = \frac{\lambda^2}{2(4\pi)^2} \left(\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0)\right). \tag{3.2.49}$$

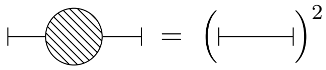
3.3 Interpretation of the divergence: the renormalized 2-point function

Let's return to the perturbative expansion of $\tilde{G}_2$:

$\tilde{G}_2(p^2, m^2) =$



where we have:



$\Sigma(p^2, m^2)$

and:

$$\Sigma(p^2, m^2) = \sum_{L=1}^{\infty} \Sigma^{(L)}(p^2, m^2) \tag{3.3.1}$$

is the self-energy containing all not facturisable diagrams. This way we obtain:

$$\tilde{G}_2(p^2, m^2) = \frac{i}{D(p.m)} \left(1 + \frac{\Sigma}{D(p,m)} + \frac{\Sigma^2}{D(p,m)^2} + \dots \right) \tag{3.3.2}$$

$$= \frac{i}{D(p,m)} \frac{1}{1 - \frac{\Sigma(p^2,m^2)}{D(p,m)}} \tag{3.3.3}$$

re-summing to all order (Dyson):

$$\tilde{G}_2(p^2, m^2) = \frac{i}{p^2 - m^2 - \Sigma(p^2, m^2) + i\emptyset^+}. \tag{3.3.4}$$

So the self-energy, containing the UV divergence, can be interpreted as a *mass shift* to the tree-level propagator. $\Sigma(p^2, m^2)$ shifts the propagator's pole.

It is therefore possible to absorb the divergence into a re-defined mass parameter, to fix finite terms:

$$m_R^2 = m^2 - \delta m^2(\kappa) \tag{3.3.5}$$

where κ is the *scheme dependence* and cancel the divergence in Σ. We have computed $\delta m^2(\kappa)$ up to one-loop:

$$\delta m^2(\kappa) = \sum_{L=1}^{\infty} \delta m^{2\,(L)}(\kappa) \tag{3.3.6}$$

and:

$$m^2 + \Sigma(p^2, m^2) = m_\kappa^2 - \delta m^{2\,(1)}(\kappa) + \Sigma^{(1)}(p^2, m_\kappa^2) + \mathcal{O}(\lambda^4) \tag{3.3.7}$$

where:

$$\delta m^{2\,(1)}(\kappa) - \Sigma^{(1)}(p^2, m^2) = \text{UV finite} \tag{3.3.8}$$

$$\implies \qquad \delta^{2\,(1)}(\kappa) = \frac{\lambda^2}{2(4\pi)^2}\left(\frac{1}{\epsilon} + \kappa\right) \tag{3.3.9}$$

and in order to make a physical prediction we must fix κ, so we must define a *renormalization scheme.*

We must also address the mass dimension of our loop corrections. There is a new dimension hidden inside the coupling in dimensional regularisation. Recall:

$$\frac{[\lambda]}{[m]} = \left(d - 3\left(\frac{d-2}{2}\right)\right) \tag{3.3.10}$$

$$\underset{d=4-2\epsilon}{=} (1+\epsilon) \tag{3.3.11}$$

so we can make the scale explicit by using:

$$\lambda_R \mu_R^\epsilon = \lambda \tag{3.3.12}$$

where:

$$[\lambda_R] = [m] \qquad , \qquad [\mu_R] = [m] \tag{3.3.13}$$

same as dimension of *bare* coupling in $d = 4$. The finite, *renormalized*, 2-point correlation function is therefore (taking $\epsilon \to 0$):

$$\tilde{G}_{2,R} = -i\Gamma_{2,R}^{-1} \tag{3.3.14}$$

where:

$$\Gamma_{2,R}(p^2, m_r^2; \kappa, \mu_R^2, \lambda_R^2) = p^2 - m_R^2 + i\emptyset^+ + \\ + \frac{\lambda_R^2}{2(4\pi)^2}\left(2 + \log(4\pi) - \gamma_E - \kappa + \log\left(\frac{\mu_R^2}{m_R^2}\right) - \beta\log\left(-\frac{\beta_+}{\beta_-}\right)\right) + \mathcal{O}(\lambda_R^4) \tag{3.3.15}$$

where we can still choose a specific value, so a renormalization scheme, for κ.

3.3.1 Renormalization of the lagrangian and counter-terms

We may apply the change of parameters already at the level of the lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda}{3!}\phi^3 \tag{3.3.16}$$

where we have now specified m_0 and λ_0 ad *bare* quantities. After the transformation:

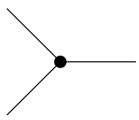
$$m_0^2 = m_R^2 + \delta m^2 \tag{3.3.17}$$
$$\lambda_0 = \lambda_R \mu_R^\epsilon \tag{3.3.18}$$

we see:

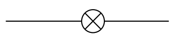
$$\mathcal{L} = \mathcal{L}\Big|_{\substack{m_0\to m_R\\ \lambda_0\to\lambda_R\mu^\epsilon}} - \frac{1}{2}\delta m^2\phi^2 \tag{3.3.19}$$

where, computing with Feynman rules, the first term gives us the same contribute as before:

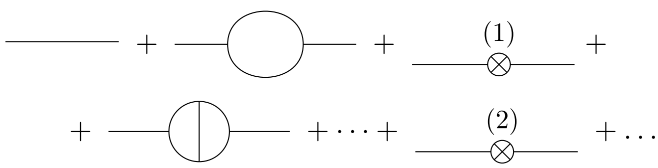
 $= \dfrac{i}{D(p, m_R)}$

 $= -i\lambda_R\mu_R^\epsilon$

but the second term gives us:

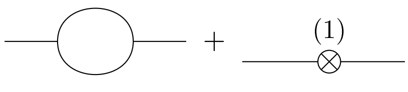
 $= -i\delta m^2$

which is the *UV counter-term from local operator.* Hence we generate renormalized (finite) quantities order by order:

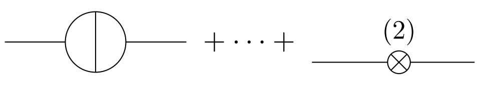


where we identify the order:

 $= \mathcal{O}(\lambda_R^0)$



$= \mathcal{O}(\lambda_R^2)$



$= \mathcal{O}(\lambda_R^4)$.

3.3.2 Renormalization schemes

We see that our finite $\tilde{G}_{2,R}$ still has an ambiguity that must be fixed before the procedure is predictive. The parameter m_0 is not the physical mass but we must specify how m_R is connected to a physical quantity, which will fix the scheme, so κ and μ_R.

We define the physical mass m_p to be at the pole of the 2-point function:

$$\tilde{G}_{2,p} = \frac{i}{p^2 - m_p^2 + i\emptyset^+} \tag{3.3.20}$$

and now consider options for the scheme.

1. Minimal subtraction (MS) Only poles from divergent graphs retained, so $\kappa_{MS} = 0$. We can then state that:

$$-iG_{2R}^{-1} = \Gamma_{2,\text{phys}} \tag{3.3.21}$$

with:

$$\lim_{p^2 \to m_{\text{phys}}^2} \Gamma_{2,\text{phys}} = i\emptyset^+ \tag{3.3.22}$$

which we match to the renormalized function:

$$\Gamma_{2,MS}(p^2, m_R^2; \mu_R^2, \lambda_R^2) \tag{3.3.23}$$

as follows:

$$\lim_{p^2 \to m_{\text{phys}}^2} \Gamma_{2,MS}(p^2, m_R^2; \mu_R^2, \lambda_R^2) = i\emptyset^+ \tag{3.3.24}$$

where, from before we have (3.3.15), so:

$$\begin{aligned} m_{\text{phys}}^2 - m_R^2 + \frac{\lambda_R^2}{2(4\pi)^2}\Bigg(2 + \log(4\pi) - \gamma_E + \log\left(\frac{\mu_R^2}{m_R^2}\right) - \\ - \beta_{\text{phys}} \log\left(-\frac{\beta_{\text{phys}+}}{\beta_{\text{phys}-}}\right)\Bigg) = \mathcal{O}(\lambda_R^4) \end{aligned} \tag{3.3.25}$$

where we use:

$$\beta_{\text{phys}} = \sqrt{1 - \frac{4m_R^2}{m_{\text{phys}}^2}} \quad , \quad \beta_{\text{phys}\pm} = \frac{1}{2}\left(1 \pm \beta_{\text{phys}}\right). \tag{3.3.26}$$

So we have fixed the scheme to a measurement at $p^2 = m_{\text{phys}}^2$ and found a (relatively complicated) relation between $m_{R,MS}^2$ and m_p^2. At leading order:

$$m_{R,MS}^2 = m_{\text{phys}}^2 + \mathcal{O}(\lambda_R^2) \tag{3.3.27}$$

so we simplify the relation above by replacing m_R with m_{phys} inside the terms at $\mathcal{O}(\lambda_R^2)$:

$$m^2_{R,MS} = m^2_{\text{phys}} + \frac{\lambda_R^2}{2(4\pi)^2}\Big(2 + \log(4\pi) - \gamma_E + \log\left(\frac{\mu_R^2}{m^2_{\text{phys}}}\right) - \\ - \frac{\pi}{\sqrt{3}}\Big) + \mathcal{O}(\lambda_R^4) \tag{3.3.28}$$

where, as we can see, we have:

$$\beta \log\left(-\frac{\beta_+}{\beta_-}\right)\Bigg|_{\substack{p^2=m^2_{\text{phys}}\\ m_R^2=m^2_{\text{phys}}}} = \frac{\pi}{\sqrt{3}} \tag{3.3.29}$$

2. Modified minimal subtraction ($\overline{\textbf{MS}}$) We can also include some additional constants in the choice of the κ which may help to include an additional finire constant in $\delta m^{2\,(1)}$:

$$\kappa_{\overline{MS}} = \log(4\pi) - \gamma_E \tag{3.3.30}$$

which can also be seen as a change in normalisation subtracting:

$$\frac{(4\pi)^\epsilon e^{-\epsilon\gamma_E}}{\epsilon} \tag{3.3.31}$$

rather than $1/\epsilon$. This relation between $m^2_{R,\overline{MS}}$ and m^2_{phys} is particularly useful for higher order correction, ad it is:

$$m_{R,\overline{MS}} = m^2_{\text{phys}} + \frac{\lambda_R^2}{2(4\pi)^2}\left(2 + \log\left(\frac{\mu_R^2}{m^2_{\text{phys}}}\right) - \frac{\pi}{\sqrt{3}}\right). \tag{3.3.32}$$

3. Fixed momentum subtraction (MOM) In this case we fix the 2-point function at a value[2] $p^2 = -M^2$ and impose:

$$\Gamma_{2,MOM}(-M^2, m_R^2; \kappa_{MOM}, \mu_R^2, \lambda_R^2) = -M^2 - m_R^2 + i\emptyset^+ \tag{3.3.33}$$

from which we determine:

$$\kappa_{MOM} = 2 + \log(4\pi) - \gamma_E + \log\left(\frac{\mu_R^2}{m_R^2}\right) - \beta_M \log\left(-\frac{\beta_{M+}}{\beta_{M-}}\right) \tag{3.3.34}$$

and so:

$$\Gamma_{2,MOM}(p^2, m_R^2; -M^2, \lambda_R^2) = p^2 - m_R^2 + i\emptyset^+ + \\ + \frac{\lambda_R^2}{2(4\pi)^2}\left(-\beta_R \log\left(-\frac{\beta_{R+}}{\beta_{R-}}\right) + \beta_M \log\left(-\frac{\beta_{M+}}{\beta_{M-}}\right)\right) + \mathcal{O}(\lambda_R^4). \tag{3.3.35}$$

[2] The minus sign is important, because doing so we don't worry about the branch-cut in the logaritm.

The relation to the physical mass can be determined as before:

$$\lim_{p^2 \to m_{\text{phys}}} \Gamma_{2,MOM}(p^2, m_R^2; M^2, \lambda_R) = i\emptyset^+ \tag{3.3.36}$$

so we have:

$$m_{R,MOM}^2 = m_{\text{phys}}^2 + \frac{\lambda_R^2}{2(4\pi)^2}\Bigg(-\frac{\pi}{\sqrt{3}} + \\ + \beta(-M^2, m_{\text{phys}}) \log\left(-\frac{\beta_+(-M^2, m_{\text{phys}})}{\beta_-(-M^2, m_{\text{phys}})} \right)\Bigg) + \mathcal{O}(\lambda_R^4). \tag{3.3.37}$$

4. On-shell subtraction Impose $m_R^2 = m_{\text{phys}}^2$ (so the equation (3.3.37)):

$$\lim_{p^2 \to m_R} \Gamma_{2,R}(p^2, m_R^2; \kappa_{OS}, \mu_R^2, \lambda_R^2) = i\emptyset^+ \tag{3.3.38}$$

so we have:

$$\kappa_{OS} = 2 + \log(4\pi) - \gamma_E + \log\left(\frac{\mu_R^2}{m_R^2}\right) - \frac{\pi}{\sqrt{3}}. \tag{3.3.39}$$

NB. For theories like QCD where m_{phys} is not in the perturbative regime we cannote use the on-shell scheme.

3.4 Complete renormalization of ϕ^3 in 4d

Having completed our analysis of the two point function we need to consider all other potentially divergent graph. The result so far for the lagrangian in terms of renormalized mass and coupling is:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_R^2\phi^2 - \frac{\lambda_R\mu_R^\epsilon}{3!}\phi^3 + \frac{1}{2}\delta m^2\phi^2 \tag{3.4.1}$$

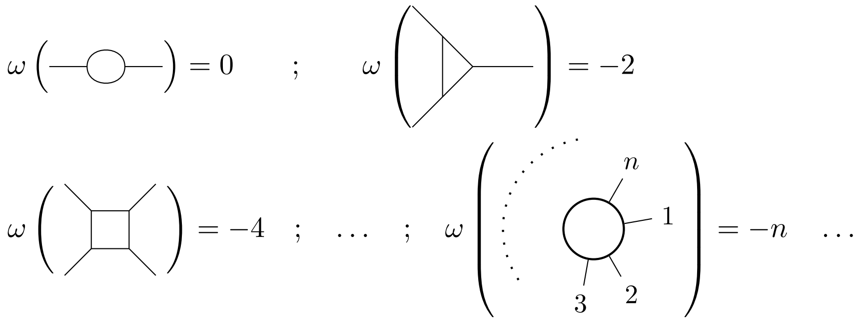
where:

$$\delta m^2 = \frac{\lambda_R^2}{2(4\pi)^2}\left(\frac{1}{\epsilon} + \kappa\right) + \mathcal{O}(\lambda_R^4) \tag{3.4.2}$$

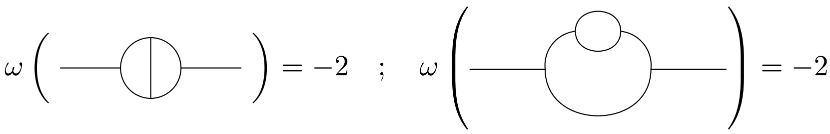
which cancel the divergence in $\tilde{G}_2$, so that $\tilde{G}_{2,R}$ is finite. In $d = 4$ we know we have $[\lambda_R] = [m]$ so we only have a finite number of divergent graph. Remember:

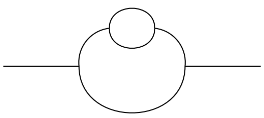
$$\omega(\Gamma) = 4 - E - V \tag{3.4.3}$$

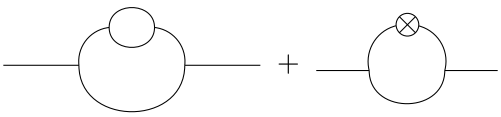
we have, at **1-loop**:



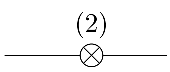
so no further 1-loop divergences. At **2-loop**:



but  has one-loop divergent subgraph which is cancelled exactly by the counter-term insertion into the one-loop graph:

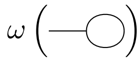
 = finite (3.4.4)

therefore we can determine that $\delta m^{2\,(2)} = 0$, so:

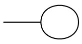
 = 0 however.

This pattern continues, to all-loop orders, so the one-loop counter-term is sufficient to render all correlation function finite, so we find δm^2 is complete at one-loop.

There is one subtlety, which is:

 $= 4 - 1 - 1 = 2$ (3.4.5)

so the tadpole graph is also divergent:



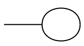
$$
\begin{aligned}
&= -\frac{i\lambda_R}{2}\mu_R^2 \int_k \frac{i}{D(k, m_R)} &(3.4.6)\\
&= \frac{\lambda_R}{2}\mu_R^{\epsilon} I_1^{(1)[4-2\epsilon]} &(3.4.7)\\
&= -i\frac{\lambda_R}{2}\frac{\Gamma(-1+\epsilon)}{\Gamma(1)}\frac{1}{(4\pi)^{2-\epsilon}}(m_R^2)^{1-\epsilon}\mu_R^{\epsilon} &(3.4.8)\\
&= \frac{i\lambda_R \mu_R^{-\epsilon m_R^2}}{2(4\pi)^2}\frac{\Gamma(1+\epsilon)}{(1-\epsilon)\epsilon}\left(\frac{\mu_R^2}{m_R^2}\right)^{\epsilon} &(3.4.9)\\
&= i\frac{\lambda_R}{2(4\pi)^2}m_R^2\left(\frac{1}{\epsilon + \mathcal{O}(\epsilon^0)}\right) &(3.4.10)
\end{aligned}
$$

where we applied the relation $\Gamma(z+1) = z\Gamma(z)$. This graph contributes to the vacuum expectation value:

$$\langle 0|\,\phi\,|0\rangle_{\text{free}} = 0 \tag{3.4.11}$$

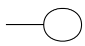
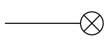
and we know:

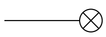
$$\langle 0|\,\phi\,|0\rangle_{\text{int}} = \langle 0|\,\phi\, S_{\text{int}}(\phi)\,|0\rangle_{\text{free}} + \dots \tag{3.4.12}$$

$=$  $+\dots$ (3.4.13)

The vacuum energy represents a physical observable and therefore we may renormalize the divergence into the vacuum. We would like to:

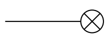
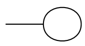
$$
\begin{aligned}
\langle 0|\,\phi\,|0\rangle_R &= \langle\phi\rangle_{\text{phys}} &(3.4.14)\\
&= 0 &(3.4.15)
\end{aligned}
$$

$=$  $+$  $= 0$ (3.4.16)

where we have  the tadpole counter-term to cancel completly the divergence. The counter-term is introduced as a linear term into the lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_R^2\phi^2 - \frac{\lambda_R\mu_R^{\epsilon}}{6}\phi^3 - \frac{1}{2}\delta m^2\phi^2 + \delta\langle\phi\rangle\phi \tag{3.4.17}$$

where:

 $=$  $= \delta\langle\phi\rangle$ (3.4.18)

in order to cancel the quantum corrections exactly.

3.5 Alternative regularisation scheme

Historically, dimensional regularization was introduced in 1972 (from ’t Hoodt, Veltman and Giandiagi, Bollini). Other options are listed below.

Cut-off regularisation (e.g. Schwinger, Feynman etc.) Wick rotation and Feynman parametrisation work in the same way. We have:

$$I_2^{(1)} = \int_k^\Lambda \frac{1}{D(k,m)D(k-p,m)} \tag{3.5.1}$$

$$= \int_0^1 \mathrm{d}\alpha \int^\Lambda \frac{\mathrm{d}^4 k}{(k^2-\Delta)^2} \tag{3.5.2}$$

$$= i\int_0^1 \mathrm{d}\alpha \int_0^\Lambda \frac{\mathrm{d}|k_E|\,|k_E|^3}{(k_E^2+\Delta)^2}\frac{2\pi^2}{(2\pi)^4} \tag{3.5.3}$$

$$= i\int_0^1 \mathrm{d}\alpha \frac{1}{16\pi^2}\int_0^{\Lambda^2} \mathrm{d}y\, \frac{y}{(y+\Delta)^2} \tag{3.5.4}$$

$$\approx \frac{i}{16\pi^2}\int_0^1 \mathrm{d}\alpha \left(\log\left(\frac{\Lambda^2}{\Delta}\right)-1\right) \tag{3.5.5}$$

where in (3.5.3) we perform the Wick rotation and compute the integral for the angular coordinate.

Lattice regularisation Discretizing space-time (minimum spacing implies maximum energy) also regulates UV divergences. We can keep gauge invariance but break Poincaré invariance. If we write:

$$I_2^{(1)[d]}(p^2,m^2) = \int_k \left(\frac{1}{D(k,m)D(k-p,m)} - \frac{1}{D^2(k,\mu)} + \frac{1}{D^2(k,\mu)}\right) \tag{3.5.6}$$

by subtracting at fixed scale μ then the divergent integral:

$$\int_k \frac{1}{D^2(k,\mu)} \propto \log\left(\frac{1}{\mu^2\alpha^2}\right) \tag{3.5.7}$$

where $x^\mu = n^\mu a$, where we indicate a like the lattice spacing, and we have $-\pi/a \leq k^\Lambda \leq \pi/a$.

Pauli-Villars regularisation In this method we introduce new, large mass, states such the each propagator becomes:

$$\frac{1}{D(p,m)} \longrightarrow \frac{1}{D(p,m)} - \frac{1}{D(p,M)} \tag{3.5.8}$$

$$= \frac{m^2-M^2}{D(p,m)D(p,M)}. \tag{3.5.9}$$

The additional propagator factors enter the Feynman parameters and regulate the integrals but all expression depend on M. In facts the additional power of momentum downstairs give us faster UV convergence, and we can still proceed with Feynman parameters, so keep integral in 4d, and they wil depend on M.

For QED, with this method, we can maintain gauge invariance.

Analytic regularisation (see Bogolinbov, Zinnermann etc.) We start from the observation that:

$$\frac{\partial}{\partial p^2}\Sigma^{(1)[4]}(p^2,m^2) \tag{3.5.10}$$

is finite. From here we can begin to differentiate under the integral. We have:

$$\frac{\partial}{\partial p^2} = \frac{p^\mu}{2p^2}\frac{\partial}{\partial p^\mu} \tag{3.5.11}$$

$$\frac{\partial}{\partial p^\mu}\frac{1}{D(k-p,m)} = \frac{(k-p)_\mu}{D(k-p,m)^2} \tag{3.5.12}$$

$$\implies \quad \frac{\partial}{\partial p^\mu}\Sigma^{(1)[4]}(p^2,m^2) = \frac{-i\lambda^2}{2}\frac{1}{2p^2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{p\cdot(k-p)}{D(k,m)D(k-p,m)^2}. \tag{3.5.13}$$

The additional powers of k downstairs aresufficient to regulate the divergence in the UV. We can perform the Feynman parametrisation via:

$$\frac{1}{A^2B} = 2\int_0^1 \mathrm{d}\alpha\,\alpha\frac{1}{(\alpha A+(1-\alpha)B)^3} \tag{3.5.14}$$

which bring us to:

$$\int_k \frac{p\cdot(k-p)}{D(k,m)D(k-p,m)^2} = 2\int_0^1\mathrm{d}\alpha\,\alpha\int_k\frac{p\cdot(k-p)}{((k-\alpha p)^2-\Delta)^3} \tag{3.5.15}$$

with:

$$\Delta = -\alpha(1-\alpha)p^2+m^2 \tag{3.5.16}$$

and so:

$$\int_k \frac{p\cdot(k-p)}{D(k,m)D(k-p,m)^2} = 2\int_0^1\mathrm{d}\alpha\,\alpha\int_k\frac{p\cdot(k+\alpha p-p)}{(k^2-\Delta)^3}. \tag{3.5.17}$$

Now rewrite:

$$\frac{\partial}{\partial p^2}\Sigma^{(1)[4]} \tag{3.5.18}$$

so we get:

$$\frac{\partial}{\partial p^2}\Sigma^{(1)[4]}(p^2,m^2) = \frac{-i\lambda^2}{2}\frac{1}{p^2}\int_0^1\mathrm{d}\alpha\,\alpha\int_k\frac{k\cdot p-(1-\alpha)p^2}{(k^2-\Delta)^3} \tag{3.5.19}$$

but we have an antysimmetric integrand over a symmetric integration range, so:

$$\int_k\frac{k\cdot p}{(k^2-\Delta)^n} = 0 \tag{3.5.20}$$

and we obtain:

$$\frac{\partial}{\partial p^2}\Sigma^{(1)[4]} = \frac{i\lambda^2}{2}\int_0^1 \mathrm{d}\alpha\, \alpha(1-\alpha)\, T_3^{[4]} \tag{3.5.21}$$

$$= \frac{i\lambda^2}{2}\int_0^1 \mathrm{d}\alpha\, \alpha(1-\alpha)\left(\frac{-i}{(4\pi)^4}\frac{1}{2\Delta}\right) \tag{3.5.22}$$

$$= -\frac{\lambda^2}{4}\frac{1}{(4\pi)^2}\frac{\partial}{\partial p^2}\int_0^1 \mathrm{d}\alpha\, \log\left(-\alpha(1-\alpha)p^2 + m^2\right) \tag{3.5.23}$$

where we revrite $1/\Delta$ as the derivative of a logaritm. In the end we have that the self energy is equal to a finite integral:

$$\Sigma^{(1)[4]} = -\frac{\lambda^2}{4(4\pi)^2}\int_0^1 \mathrm{d}\alpha\, \log(\Delta) + C \tag{3.5.24}$$

where in C (the constant that contains the boundary value) we have the divergence. We can fix the constant using the renormalization scheme condition, e.g. thew *on-shell scheme*:

$$\Sigma^{(1)[4]}(m^2, m^2) = 0 \tag{3.5.25}$$

so we get:

$$\Sigma^{(1)[4]}(p^2, m^2) = -\frac{\lambda^2}{4(4\pi)^2}\left(\int_0^1 \mathrm{d}\alpha\, \log(\Delta) - \int_0^1 \mathrm{d}\alpha\, \log\left(\Delta\Big|_{p^2=m^2}\right)\right) \tag{3.5.26}$$

$$= \frac{-\lambda^2}{4(4\pi)^2}\int_0^1 \mathrm{d}\alpha\, \log\left(\frac{1-\alpha(1-\alpha)\frac{p^2}{m^2}}{1-\alpha(1-\alpha)}\right). \tag{3.5.27}$$

3.6 Renormalization of ϕ^3 in 6d at one-loop

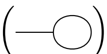
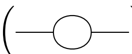
In 6 dimension the coupling constant, λ, is dimensionless. The theory is renormalizable, but in a more intricate way than in $4d$. We start with the bare lagrangian:

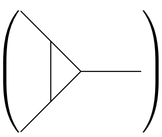
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{3!}\phi_0^3 \tag{3.6.1}$$

in which all quantities are denoted a bare using a subscript zero. The superficial degree of divergence is:

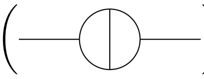
$$\omega(\Gamma) = 6 - 2E \tag{3.6.2}$$

so:

ω () $= 4$; ω () $= 2$

ω () $= 0$ and < 0 for $E > 0$.

This means we must also take care of divergence in the 3-point vertex function. Since ω is independent from the number of vertices (V) the higher loop corrections to the 1-, 2- and 3- point functions also diverge (not just subgraphs), e.g.:

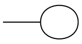
$$\omega\left(\right.$$  $$\left.\right) = 2. \tag{3.6.3}$$

We will first compute the divergent graphs and then show how they may be absorbed into a redefinition of the fields and parameters in the lagrangian.

3.6.1 Divergent graphs at 1-loop

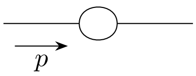
In this section we will compute directly using the bare lagrangian.

Let's start with the **todpole**:



$$\begin{aligned} &= -\frac{i\lambda_0}{2} iT_1^{[6-2\epsilon]}(m_0^2) && (3.6.4)\\ &= \frac{\lambda_0}{2}\frac{-i}{(4\pi)^{3-\epsilon}}(m_0^2)^{2-\epsilon}\frac{\Gamma(-2+\epsilon)}{\Gamma(1)} && (3.6.5)\\ &= -\frac{i\lambda_0}{2(4\pi)^3}\frac{(4\pi)^3\Gamma(1+\epsilon)m_0^4(m_0^2)^{-\epsilon}}{\epsilon(1-\epsilon)(2-\epsilon)} && (3.6.6)\\ &= -i\frac{\lambda_0}{2}\frac{1}{2(4\pi)^3}m_0^4\left(\frac{1}{\epsilon}+\frac{3}{2}-\gamma_E+\log(4\pi)-\log(m_0^2)+\mathcal{O}(\epsilon)\right). && (3.6.7)\end{aligned}$$

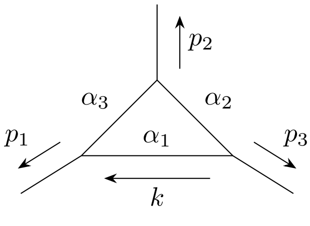
We can analize the **bubble** (where we use the Wick rotation and the Feynman parametrisation):



$$\begin{aligned} &= -\frac{(i\lambda_0)^2}{2}\int_0^1 \mathrm{d}\alpha\, i^2 T_2^{[6-2\epsilon]}\left(-\alpha(1-\alpha)p^2+m_0^2\right) && (3.6.8)\\ &= \frac{i\lambda_0^2}{2(4\pi)^3}\left(\frac{1}{\epsilon}\left(-m_0^2+\frac{p^2}{6}\right)+\mathcal{O}(\epsilon^0)\right) && (3.6.9)\end{aligned}$$

for the term $\mathcal{O}(\epsilon^0)$ see Mathematica. Now we can see that the divergence has a kinematic dependence, so changing dimension change the kinematic dependence.

For the **triangle** we have:



$$= (-i\lambda_0)^3\int_k \frac{i^3}{D(k,m)D(k-p_1,m)D(k-p_1-p_2,m)} \tag{3.6.10}$$

we can use the relation (3 point function, so 3 Feynman parameters):

$$\frac{1}{ABC} = \int \prod \mathrm{d}\alpha_i \frac{2\delta(1-\alpha_1-\alpha_2-\alpha_3)}{(\alpha_1 A + \alpha_2 B + \alpha_3 C)^3} \tag{3.6.11}$$

so we have:

$$= \lambda_0^3 \int_0^1 \mathrm{d}\alpha_1 \int_0^{1-\alpha_1} \mathrm{d}\alpha_2 \int_k \frac{2}{(k^2-\Delta_3)^3} \tag{3.6.12}$$

where we write:

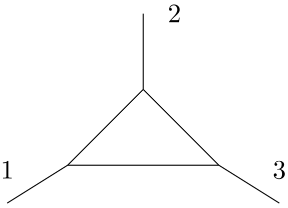
$$\Delta_3 = m_0^2 - (1-\alpha_1)\alpha_1 p_1^2 - (1-\alpha_2)\alpha_2 p_2^2 - 2\alpha_1\alpha_2 p_1 \cdot p_2 \tag{3.6.13}$$

so:

$$= -i\lambda_0^3 2 \int_0^1 \mathrm{d}\alpha_1 \int_0^{1-\alpha_1} \mathrm{d}\alpha_1\, T_3^{[6-2\epsilon]}(\Delta_3) \tag{3.6.14}$$

$$= (-i\lambda_0)(-i\lambda_0^2) \frac{2\Gamma(\epsilon)}{(4\pi)^{3-\epsilon}\Gamma(3)} \int_0^1 \mathrm{d}\alpha_1 \int_0^{1-\alpha_1} \mathrm{d}\alpha_2\, \Delta_3^{-\epsilon} \tag{3.6.15}$$

expanding up to $\mathcal{O}(\epsilon^0)$ is quite difficult in closed form (polylogarithms with complicated avgs) keeping just $1/\epsilon$ is quite simple:

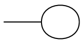
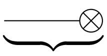


$$= \frac{(-i\lambda_0)\lambda_0^2}{2(4\pi)^3}\left(\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0)\right) \tag{3.6.16}$$

and again see Mathematica.

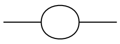
3.6.2 Counter-terms and the renormalized lagrangian

The *todpole divergence* is absorbed into the vacuum expectation value as in the $d = 4$ case, imposed by introducing a linear counter-term matching the todpole graph:

 +  $= 0$ (3.6.17)

$\propto \delta\langle\phi\rangle$

where the counter term introduces a linear term in $\mathcal{L}$. The *bubble divergence* has changed it's functional form and is no longer absorbed by a simple mass shift (because now we have a kinematic dependence). We saw:

 $\sim \left(-m_0^2 + \frac{p^2}{6}\right)\frac{1}{\epsilon}.$ (3.6.18)

The first term can be obtained via a redefinition of the mass and a mass counter-term $\sim \delta m^2 \phi^2$. The second term can be obtained from a redefinition of the field ϕ and a counter-term $\sim \partial_\mu\phi\partial^\mu\phi\delta\phi$. These re-definition are called *mass* and *wave-function renormalization* respectively. This can be written clearly introducing *renormalization constant* Z_m and Z_ϕ:

$$m_0^2 = Z_m m_R^2 \tag{3.6.19}$$

$$\phi_0 = \sqrt{Z_\phi}\phi_R \tag{3.6.20}$$

where:

$$\delta_x = Z_x - 1 \qquad , \quad \text{for } x = m, \phi. \tag{3.6.21}$$

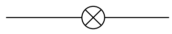
Applying this transformation to the lagrangian leads to:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_R\partial^\mu\phi_R - \frac{1}{2}m_R^2\phi_R^2 + \mathcal{L}_{int} + \underbrace{(1 - z_\phi)}_{\delta\phi=\delta_2}\frac{1}{2}\partial_\mu\phi_R\partial^\mu\phi_R - \\ - \underbrace{(1 - Z_m Z_\phi)}_{\delta_3}\frac{1}{2}m_R^2\phi_R^2 \tag{3.6.22}$$

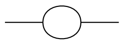
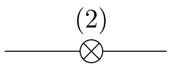
from which we obtain, renaming:

$$\delta_\phi = \delta_2 \qquad ; \qquad \delta_3 = (Z_m Z_\phi - 1)$$

a counter-term Feynman rule:

 $= i\left(\delta_2 p^2 - \delta_3 m_R^2\right).$ (3.6.23)

The renormalization constant can then be fixed at 1-loop via:

 +  = UV finite (3.6.24)

for all values of p^2 and m_R^2.

We still have to deal with $\mathcal{L}_{int}$ and the renormalization of the coupling and absorbing the vertex divergence.

The vertex divergence The divergence in the triangle function requires another counter-term that can be obtained through a redefinition of the coupling:

$$\lambda_0 = Z_\lambda \lambda_R \mu_R^\epsilon \qquad , \qquad \frac{[\lambda_R]}{[m]} = 0 \tag{3.6.25}$$

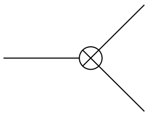
where μ_R is necessary to keep λ_R dimensionless. So we have:

$$\mathcal{L}_{int} = \frac{1}{3!}\lambda_0\phi_0^3 \tag{3.6.26}$$

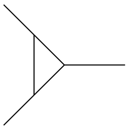
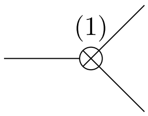
$$= \frac{\lambda_R\mu_R^\epsilon}{3!}\underbrace{Z_\lambda Z_\phi^{3/2}}_{=Z_1=1+\delta_1}\phi_R^3 \tag{3.6.27}$$

$$= \frac{\lambda_R\mu_R^\epsilon}{3!}(1 + \delta_1)\phi_R^3. \tag{3.6.28}$$

This is *charge renormalization* and leads to a new counter-term with Feynman rule:

 $= -i\lambda_R\mu^\epsilon\delta_1$ (3.6.29)

which can be fixed (at one-loop) using:

 +  = UV finite. (3.6.30)

NB. We have $\delta_3 = \delta_m + \delta_\phi + \delta_m\delta_\phi$, using $\delta_\phi = \delta_2$, but the product term is always of higher order in perturbation theory than the linear terms so:

$$\delta_m^{(1)} = \delta_3^{(1)} - \delta_\phi^{(1)} \tag{3.6.31}$$

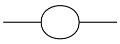
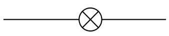
$$\delta_m^{(2)} = \delta_3^{(2)} - \delta_\phi^{(2)} - \delta_3^{(1)}\delta_\phi^{(1)} \tag{3.6.32}$$

we also have that:

$$Z_1 = Z_\lambda Z_\phi^{3/2} \quad \Longrightarrow \quad \delta_\lambda^{(1)} = \delta_1^{(1)} - \frac{3}{2}\delta_\phi^{(1)}. \tag{3.6.33}$$

3.6.3 ($\overline{\textbf{MS}}$) renormalization constants at 1-loop

We know have all the ingradients we need to complete the computation of δ_ϕ, δ_m and δ_λ. Explicitly:

 +  = UV finite. (3.6.34)

that implies:

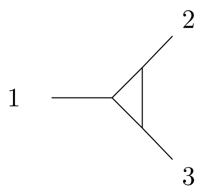
$$\frac{i\lambda_R^2\mu_R^{2\epsilon}}{2(4\pi)^3}\underbrace{\frac{(4\pi)^\epsilon\Gamma(1+\epsilon)}{\epsilon}}_{=\Gamma_\Gamma}\left(-m_R^2 + \frac{p^2}{6}\right) + i\left(\delta_2^{(1)}p^2 - \delta_3^{(1)}m_R^2\right) = 0 \tag{3.6.35}$$

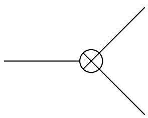
so we have:

$$\delta_2^{(1)} = -\frac{\lambda_R^2\mu_R^{2\epsilon}}{2(4\pi)^3}\sigma_\Gamma\frac{1}{6\epsilon} \tag{3.6.36}$$

$$\delta_3^{(1)} = -\frac{\lambda_R^2\mu_R^{2\epsilon}}{2(4\pi)^3}\sigma_\Gamma\frac{1}{\epsilon} \tag{3.6.37}$$

and:



+  = UV finite (3.6.38) which implies:

$$(-i\lambda_R\mu_R^\epsilon)\frac{\lambda_R^2\mu_R^{2\epsilon}}{2(4\pi)^3}\Gamma_\Gamma\left(\frac{1}{\epsilon}\right)+(-i\lambda_R\mu_R^\epsilon)\delta_1^{(1)}=0 \tag{3.6.39}$$

which gives us:

$$\delta_1^{(1)}=-\frac{\lambda_R^2\mu_R^{2\epsilon}}{2(4\pi)^3}\sigma_\Gamma\left(\frac{1}{\epsilon}\right). \tag{3.6.40}$$

Back in terms of $\delta_\phi^{(1)}, \delta_m^{(1)}$ and $\delta_\lambda^{(1)}$ we have (with $N_\lambda=\lambda_R^2\mu_R^{2\epsilon}\sigma_\Gamma/2(4\pi)^3$):

$$N_\lambda^{-1}\delta_\phi^{(1)}=-\frac{1}{6\epsilon} \tag{3.6.41}$$

$$N_\lambda^{-1}\delta_m^{(1)}=-\frac{1}{\epsilon}+\frac{1}{6\epsilon}=-\frac{5}{6\epsilon} \tag{3.6.42}$$

$$W_\lambda^{-1}\delta_\lambda^{(1)}=-\frac{1}{\epsilon}-\frac{3}{2}\left(-\frac{1}{6\epsilon}\right)=-\frac{3}{4\epsilon}. \tag{3.6.43}$$

3.6.4 Other renormalization schemes

For on-shell and momentum subtraction (MOM) we impose a stricter contidion that determined the finite terms. Since there are now 3 renormalization constants we must find an extendeded set of condition. These are:

$$\lim_{p^2\to x}\Gamma_{2,R}(p^2,m_R^2;k_2,k_3,\mu_R^2,\lambda_R^2)=i\emptyset^+ \tag{3.6.44}$$

$$\lim_{p^2\to x}\frac{\partial\Gamma_{2,R}}{\partial p^2}=1 \tag{3.6.45}$$

with $x=-M^2$ for MOM and $x=m_{phys}^2=m_R^2$ for the on-shell scheme.

In MOM and $\overline{MS}$ schemes we must also derive the relation between m_R^2 and m_{phys}^2. For k, we impose (with the minus sign for convetion):

$$\lim_{\substack{p_i\to\text{value where}\\ \lambda_{\text{phys}}\text{ measured}}} G_{3,R}(p_1,p_2;k_1,\mu_R^2,m_R^2,\lambda_R^2)=-\lambda_{\text{phys}} \tag{3.6.46}$$

for the on-shell scheme and (for example):

$$\lim_{p_i\to p_{sym}} G_{3,R}(p_1,p_2;k_1,\mu_R^2,m_R^2,\lambda_R^2)=-\lambda_R \tag{3.6.47}$$

for the MOM scheme (no quantum corrections to the vertex at $p_i=p_{sym}$). The symmetric point is defined as:

$$p_i^2=-\frac{2}{3}M^2 \quad\Longrightarrow\quad p_i\cdot p_j=\frac{M^2}{3}\quad\text{for } i\neq j \tag{3.6.48}$$

e.g.:

$$p_3^2=(p_1+p_2)^2=p_1^2+p_2^2+2p_1\cdot p_2. \tag{3.6.49}$$

3.7 Renormalization scale dependence

Having fixed the 3 scheme constant k_i with the conditions on Γ_2, $\frac{\partial}{\partial p}\Gamma_2$ and G_3, we have determined m_R and λ_R in terms of same fixed values (physical measurements) m_{phys} and λ_{phys} but the dependence on μ_R^2 remains. This is a feature of the re-organized renormalized perturbative series. Eventually it must be that the dependences goes away as we sum to all orders - the lagrangian after all has no changed:

$$\mathcal{L}(\phi_0, m_0, \lambda_0) = \mathcal{L}(\phi_R, m_R, \lambda_R; \mu_R) \tag{3.7.1}$$

which means the bare correlation functions do not depend on μ_R do not depend on μ_R:

$$\tilde{G}_0(p_1, \dots, p_n; \lambda_0) = FT\left(\langle 0| T[\phi_0(x_1)\phi_0(x_2)\dots] |0\rangle\right) \tag{3.7.2}$$

$$= Z_\phi^{n/2}(\mu_R, \lambda_R(\mu_R)) FT(\langle 0| T[\phi_0(x_1)\phi_0(x_2)\dots] |0\rangle) \tag{3.7.3}$$

$$= Z_\phi^{n/2}(\mu_R, \lambda_R(\mu_R)) \tilde{G}_R(p_1, \dots, p_n; \lambda_R(\mu_R), \mu_R) \tag{3.7.4}$$

where with FT we indicate the Fourier transform. So the μ_R dependence of $\tilde{G}_R$ must cancel exactly with the μ_R dependence of $Z_\phi^{n/2}$. In other words:

$$0 = \frac{\partial}{\partial \mu_R}\tilde{G}_0 = \frac{\partial}{\partial \mu_R}\left(Z_\phi^{n/2}\right)\tilde{G}_R + Z_\phi^{n/2}\frac{\partial}{\partial \mu_R}\tilde{G}_R. \tag{3.7.5}$$

We will explore the conseguence of this scale dependence when studying the *renormalization group.*

Chapter 4

Loop integration methods in dimensional regularization

In this chapter will expand on the techniques used in chapter §3 and develop methods need to continue our discussion to gauge theories.

We will study Feynman and Schwinger parameter representations of one-loop integrals in dimensional regularisation. We will found on all-loop parametrisation using F and U Symanzik polynomials. Also, we will study the tensor reduction in parameter and momentum space, to conclude defining che Clifford Algebra for $d = 4 - 2\epsilon$.

Further reading:

- *Scattering amplitudes in QFT* [chapter 4], Badger, Henn, Plefka, Zoia (Springer 2024), https://arxiv.org/pdf/2306.05976.
- *Feynman Integrals* [Comprehensive: 800+ pages], Weinzierl (Springer 2022), https://arxiv.org/pdf/2201.03593.

Note. In this chapter, as the previous one, we will represent the Feynman diagrams for scalar fields with solid lines, and not with dashed.

4.1 Feynman and Schwinger parameters

We came across 3 examples of Feynman parametrisation. We saw:

$$\frac{1}{AB} = \int_0^1 \mathrm{d}\alpha\, \frac{1}{[A\alpha + B(1-\alpha)]^2} \tag{4.1.1}$$

$$\frac{1}{AB^2} = 2\int_0^1 \mathrm{d}\alpha\, \alpha \frac{1}{[A\alpha + B(1-\alpha)]^2} \tag{4.1.2}$$

$$\frac{1}{ABC} = 2\int_0^1 \mathrm{d}\alpha_1 \int_0^{1-\alpha_1} \mathrm{d}\alpha_2 \frac{1}{[A\alpha_1 + B\alpha_2 + C(1-\alpha_1-\alpha_2)]^3} \tag{4.1.3}$$

but we can write the general version:

$$\frac{1}{\prod_{i=1}^{n} A_i^{\nu_i}} = \frac{\Gamma\left(\sum_{i=1}^{n} \nu_i\right)}{\prod_{i=1}^{n} \Gamma\left(\nu_i\right)} \int_0^1 \prod_{i=0}^{n} \mathrm{d}\alpha_i \, \alpha_i^{\nu_i - 1} \frac{\delta(1 - \sum_i \alpha_i)}{\left(\sum_i D_i \alpha_i\right)^{\sum_i \nu_i}} \tag{4.1.4}$$

which is valid also for non-integer ν_i. A similar transformation is provided through *Schwinger parameters*:

$$\frac{1}{D^{\nu}} = \frac{(-1)^n}{\Gamma(\nu)} \int_0^{\infty} \mathrm{d}t \, t^{\nu-1} \exp\{-tD\}. \tag{4.1.5}$$

The 1-loop bubble example, so (3.2.3), becomes (performing at some point the shift $k \to k + t_2 p/(t_1 + t_2)$):

$$I_2^{(1)[d]} = \int_k \int_0^{\infty} \mathrm{d}t_1 \int_0^{\infty} \mathrm{d}t_2 \, \exp\{-t_1 D(k,m) - t_2 D(k-p,m)\} \tag{4.1.6}$$

$$= \int_k \int_{t_i} \exp\left\{-(t_1 + t_2)(k^2 - m^2) - \frac{t_1 t_2}{t_1 + t_2} p^2\right\} \tag{4.1.7}$$

$$= \int_k \int_{t_1} \exp\left\{(ak^2 + b)\right\} \tag{4.1.8}$$

where we write:

$$a = t_1 + t_2 \qquad ; \qquad b = m^2(t_1 + t_2) - \frac{t_1 t_2}{t_1 + t_2} p^2. \tag{4.1.9}$$

The integral over k can be performed after a Wick rotation:

$$I_2^{(1)[d]} = i \int_{t_i} \int_{k_E} \exp\left\{-a k_E^2\right\} \exp\{b\} \tag{4.1.10}$$

$$= \frac{i}{(4\pi)^{d/2}} \int_{t_i} \exp\{b\} a^{-d/2} \tag{4.1.11}$$

which we can see the exponential doesn't depend on the loop momentum.

4.1.1 Tensor integrals with Feynman/Schwinger parameters

This is very important in gauge theories where we will find loop momentum dependent numerators, which require further analysis. We did find one case already in the case of $\frac{\partial}{\partial p^2}\Sigma$ for ϕ^3 although it was handeled in an easy manner. Let's try the example again using Schwinger parameters; we define:

$$I_2^{(1)[d]\mu}(p,m) = \int_k \frac{k^{\mu}}{D(k,m) D(k-p,m)} \tag{4.1.12}$$

now if we perform the shift to complete the square of the argument of the exponential:

$$k \longrightarrow k + t_2 p/(t_1 + t_2) \tag{4.1.13}$$

that we used to put the denominator in standard form also affects the numerator so:

$$I_2^{(1)[d]\mu} = \int_k \int_{t_i} \left(k + \frac{t_2 p}{t_1 + t_2}\right)^\mu \exp\{ak^2 - b\} \tag{4.1.14}$$

$$= p^\mu \int_{t_i} \frac{t_2}{t_1 + t_2} \frac{i}{(4\pi)^{d/2}} a^{-d/2} \exp\{-b\} \tag{4.1.15}$$

where we performed the rotation and integrated over k_E and have used:

$$\int_k k^\mu \exp\{ak^2\} = 0. \tag{4.1.16}$$

The parametric integration seems more difficult, but we can use the symmetry[1] in $t_1 \leftrightarrow t_2$ to write:

$$I_2^{(1)[d]\mu} = \frac{p^\mu}{2} \int_{t_i} \left(\frac{t_2}{t_1 + t_2} + \frac{t_1}{t_1 + t_2}\right) \frac{i}{(4\pi)^{d/2}} a^{-d/2} \exp\{-b\} \tag{4.1.17}$$

$$= \frac{p^\mu}{2} I_2^{(1)[d]}. \tag{4.1.18}$$

We define the **tensor rank of a Feynman integral** as the total power of all loop momenta in the numerator of the integrand. As we increase the tensor rank we encouter new loop momentum integrals; some examples are:

$$\int_k \exp\{ak^2\} = \frac{i}{(4\pi)^{d/2}} a^{-d/2} = \kappa \tag{4.1.19}$$

$$\int_k k^\mu \exp\{ak^2\} = 0 \tag{4.1.20}$$

$$\int_k k^\mu k^\nu \exp\{ak^2\} = -\frac{1}{2a} g^{\mu\nu} \kappa \tag{4.1.21}$$

$$\int_k k^{\mu_1} \dots k^{\mu_{2n+1}} \exp\{ak^2\} = 0 \qquad n \in \mathbb{Z} \tag{4.1.22}$$

$$\int_k k^{\mu_1} \dots k^{\mu_{2n}} \exp\{ak^2\} = -\frac{1}{2a} \sum_{i=2}^{2n} g^{\mu_1 \mu_i} \int_k \frac{k^{\mu_1} \dots k^{\mu_{2n}}}{k^{\mu_1} k^{\mu_i}} \exp\{ak^2\} \qquad n \in \mathbb{Z}. \tag{4.1.23}$$

In (4.1.21) the metric tensor is the only tensor we can use, over performed the integration, to give tensor structure (I_2, after the shift, depends on p^2). We have this last case (4.1.23) as an exercise though it's simple to prove (4.1.21): firstly we need to identify a basis of tensor structures (after integration). There are no external momenta so there's only one tensor structure $g^{\mu\nu}$:

$$\int_k k^\mu k^\nu \exp\{ak^2\} = A g^{\mu\nu} \tag{4.1.24}$$

[1] If I associate t_1 to the other propagator and t_2 to the remaining and reperform the computation, we obtain the same prefactor but with $t_1 \leftrightarrow t_2$.

where A is a *scalar form factor*. Contracting on both sides with $g_{\mu\nu}$ and using $g^{\mu\nu}g_{\mu\nu} = g^{\mu}_{\mu} = d$ gives:

$$dA = \int_k k^2 \exp\{ak^2\} \tag{4.1.25}$$
$$= \int_k \frac{\partial}{\partial a} \exp\{ak^2\} \tag{4.1.26}$$
$$= \frac{\partial}{\partial a} \int_k \exp\{ak^2\} \tag{4.1.27}$$
$$= \frac{\partial}{\partial a} \left(\frac{i}{(4\pi)^{d/4}} a^{-d/2} \right) \tag{4.1.28}$$
$$= -\frac{d}{2}\frac{1}{a}\frac{i}{(4\pi)^{d/2}} a^{-d/2} \tag{4.1.29}$$
$$dA = -\frac{d}{2a}\kappa \tag{4.1.30}$$
$$A = -\frac{1}{2a}\kappa \tag{4.1.31}$$

as previously stated.

Similar expressions are obtained using Feynman parameters after following the same steps. For example:

$$I_2^{(1)[d]\mu} = \int_0^1 \mathrm{d}\alpha \int_k \frac{k^\mu + \alpha p^\mu}{(k^2 - \Delta)^2} \tag{4.1.32}$$
$$= p^\mu \int_0^1 \mathrm{d}\alpha\, \alpha T_2^{[d]}(\Delta) \tag{4.1.33}$$

and:

$$I_2^{(1)[d]\mu\nu} = \int_0^1 \mathrm{d}\alpha \int_k \frac{(k^\mu + \alpha p^\mu)(k^\nu + \alpha p^\nu)}{(k^2 - \Delta)^2}. \tag{4.1.34}$$

Exercise 2. Show that we may write the rank 2 tensor bubble integral as:

$$I_2^{(1)[d]\mu\nu} = p^\mu p^\nu \int_0^1 \mathrm{d}\alpha\, \alpha^2 T_2^{[d]}(\Delta) + g^{\mu\nu}\frac{1}{d}\int_0^1 \mathrm{d}\alpha \left(T_1^{[d]}(\Delta) + \Delta T_2^{[d]}(\Delta) \right) \tag{4.1.35}$$

where:

$$\Delta = -\alpha(1-\alpha)p^2 + m^2. \tag{4.1.36}$$

You can also may use the result:

$$\int_k \frac{k^\mu k^\nu}{(k^2 - \Delta)^n} = \frac{1}{d} g^{\mu\nu} \int \frac{1}{(k^2 - \Delta)^n}. \tag{4.1.37}$$

Exercise 3. Write down the general tensor decomposition for a rank-4 tensor tadpole:

$$T_n^{[d]\mu_1\ldots\mu_4}(\Delta) = \int_k \frac{k^{\mu_1}k^{\mu_2}k^{\mu_3}k^{\mu_4}}{(k^2-\Delta)^n}. \tag{4.1.38}$$

4.2 Multi-loop parametrisation and Symanzik polynomials

For the Symanzik polynomials you can see Bogner and Weinzierl's article.

It is possible to find a general parametrisation for any multi-loop graph in which we can perform the integration in k_i^μ, but leave integrals over Feynman/Schwinger parameters. Let's consider a 2-loop graph with n denominator defined through:

$$D_i = D(q_i, m_i) = q_i^2 - m_i^2 \tag{4.2.1}$$

$$q_i = \lambda_{ij}k_j^\mu + \sigma_{il}p_l^\mu \tag{4.2.2}$$

where k_j^μ are the loop momenta ($j = 1, 2$) and p_k^μ are the indipendent external momentum. We define:

$$I_n^{(2)[d]} = \int_{k_1}\int_{k_2} \frac{1}{\prod_{i=1}^n D_i}. \tag{4.2.3}$$

The combined argument of the exponential after introducing Schwinger parameters x_i for each D_i is $\sum_i x_i D_i$ which we may decompose according to the loop momentum structures.:

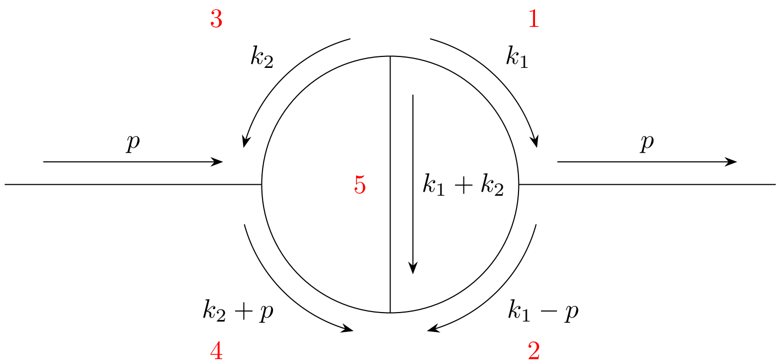
$$\sum x_i D_i = \underbrace{k_{i\mu}M_{ij}k_j^\mu}_{k^T M k} + 2\underbrace{k_{i\mu}Q_i^\mu}_{k^T Q = Q^T k} + J \tag{4.2.4}$$

where:

- M is a geometric symmetric 2x2 matrix depending on the Schwinger parameters x_i.
- Q_i^μ is a vector linear in p_l^μ with coefficient dependent on x_i.
- J is a scalar quantity depending on momentum invariants and x_i.

As a concrete example we can look at a two-loop propagator graph (e.g.

in ϕ^3 with all equal masses $m_i = m$):



The matrix M can be written simply following the propagator dependence on each Schwinger parameter:

$$M = \begin{pmatrix} x_1 + x_2 + x_3 & x_5 \\ x_5 & x_3 + x_4 + x_5 \end{pmatrix}. \tag{4.2.5}$$

Linear dependence of k_1/k_2 camess only from propagators 2 and 4 so:

$$Q^\mu = \{-p^\mu x_2, p^\mu x_4\} \tag{4.2.6}$$

and J contains all mass dependence and p^2 terms:

$$J = -\left(\sum_{i=1}^{5} x_i\right) m^2 + (x_2 + x_4)\, p^2. \tag{4.2.7}$$

From the general form (4.2.4) we can "complete the square" in k, we do that in 2 stages: first diagonalize M, then shift it using Q.

First step: A general symmetric 2x2 matrix is:

$$M = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \tag{4.2.8}$$

completing the square in k_1 can be performed via transformation with *unit determinant*:

$$\Lambda = \begin{pmatrix} 1 & -c/a \\ 0 & 1 \end{pmatrix} \tag{4.2.9}$$

where, with $\vec{k} = (k_1, k_2)$, we have:

$$\vec{k} = \Lambda \vec{k}' \tag{4.2.10}$$

so this implies:

$$\begin{aligned} \vec{k}^T M \vec{k} &= (\Lambda \vec{k}')^T M (\Lambda \vec{k}') && (4.2.11) \\ &= (\vec{k}')^T (\Lambda^T M \Lambda) \vec{k}' && (4.2.12) \\ &= (\vec{k}')^T D \vec{k}' && (4.2.13) \end{aligned}$$

with:

$$D = \Lambda^T M \Lambda = \begin{pmatrix} a & 0 \\ 0 & \frac{\det\{M\}}{a} \end{pmatrix}. \tag{4.2.14}$$

Second step: absorb the linear terms in k_i via a transformation:

$$\vec{k}' = \vec{k}'' - \Lambda^{-1} M^{-1} \vec{Q} \tag{4.2.15}$$

so the exponential's argument after stage 1 transformation:

$$\begin{aligned}
(\vec{k}')^T D \vec{k}' + 2\vec{Q}^T \Lambda \vec{k}' + J = & \qquad (4.2.16)\\
= (\vec{k}'')^T D \vec{k}'' - (\vec{k}'')^T D \Lambda^{-1} M^{-1} \vec{Q} - (\Lambda^{-1} M^{-1} \vec{Q})^T D \vec{k}'' + &\\
+ (\Lambda^{-1} M^{-1} \vec{Q})^T D \Lambda^{-1} M^{-1} \vec{Q} + &\\
+ 2\vec{Q}^T \Lambda \vec{k}'' - 2\vec{Q}^T \Lambda \Lambda^{-1} M^{-1} \vec{Q} + J & \qquad (4.2.17)\\
= \vec{k}'' D \vec{k}'' + J - \vec{Q}^T M^{-1} \vec{Q}. & \qquad (4.2.18)
\end{aligned}$$

Both steps can be combined so that:

$$\vec{k}'' = \Lambda \vec{k} - M^{-1} \vec{Q} \tag{4.2.19}$$

and the argument of the exponential is written as:

$$\prod x_i D_i = \vec{k}^T D \vec{k} + \frac{F}{u} \tag{4.2.20}$$

where:

$$u = \det\{M\} \tag{4.2.21}$$

which is the *first Symanzik polynomial*, and:

$$F = (J - \vec{Q}^T M^{-1} \vec{Q}) u \tag{4.2.22}$$

is the *second Symanzik polynomial*, and last:

$$D = \begin{pmatrix} a & 0 \\ 0 & u/a \end{pmatrix}. \tag{4.2.23}$$

The transformation has separated the loop integral so we can perfomr the integration:

$$\begin{aligned}
I_n^{(2)[d]} &= \int \prod_{i=1}^{n} \mathrm{d}x_i \, \exp\left\{\frac{F}{u}\right\} \int_{k_1} \exp\{a k_1^2\} \int_{k_2} \exp\left\{\frac{u}{a} k_2^2\right\} && (4.2.24)\\
&= \left(\frac{i}{(4\pi)^{d/2}}\right)^2 \int \prod_{i=1}^{n} \mathrm{d}x_i \, \exp\left\{\frac{F}{u}\right\} u^{-d/2}. && (4.2.25)
\end{aligned}$$

We may change from variables x_i (in $0, \infty$) to α_i (in $0, 1$) by inserting:

$$1 = \int_0^\infty \mathrm{d}\lambda \, \delta\left(\lambda - \sum_{i=1}^n x_i\right) \tag{4.2.26}$$

and rescaling each $x_i = \lambda\alpha_i$, so:

$$I_n^{(2)[d]} = \left(\frac{i}{(4\pi)^{d/2}}\right)^2 \int_0^1 \prod_{i=1}^n \mathrm{d}\alpha_i \, \delta\left(1 - \sum_{i=1}^n \alpha_i\right) \int \frac{\mathrm{d}\lambda}{\lambda} \lambda^{n-d} \exp\left\{\lambda \frac{F}{u}\right\} u^{-d/2} \tag{4.2.27}$$

using the change $z = -\lambda \frac{F}{u}$

$$= \left(\frac{i}{(4\pi)^{d/2}}\right)^2 \Gamma(n-d) \int_0^1 \prod_{i=1}^n \mathrm{d}\alpha_i \, \delta\left(1 - \sum_{i=1}^n \alpha_i\right) \left(-\frac{F}{u}\right)^{d-n} u^{-d/2} \tag{4.2.28}$$

$$= \left(\frac{i}{(4\pi)^{d/2}}\right)^2 \Gamma(n-d) \int_{\alpha_i} \delta\left(1 - \sum \alpha_i\right) \frac{u^{n-3d/2}}{(-F)^{n-d}}. \tag{4.2.29}$$

By following the same procedure we can derive a general multi-loop expression valid for arbitrary powers of the denominator.

Exercise 4. Show that a general L-loop Feynman integral with n internal lines may be written as:

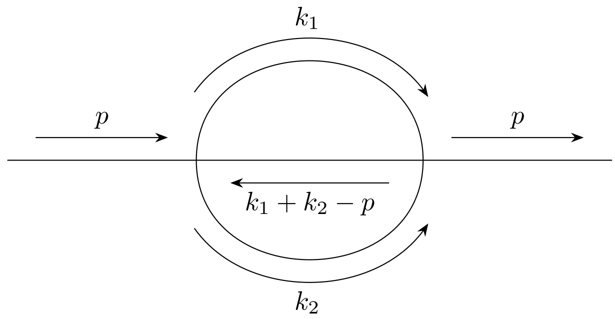
$$I_n^{(L)[d]}[\nu_1, \ldots, \nu_n] = \int \prod_{l=1}^L \frac{\mathrm{d}^d k_l}{(2\pi)^d} \frac{1}{\prod_{j=1}^n D(q_j, m_j)^{\nu_j}} \tag{4.2.30}$$

$$= \left(\frac{i}{(4\pi)^{d/2}}\right)^L \frac{\Gamma\left(\sum_{i=1}^n \nu_i - L\frac{d}{2}\right)}{\prod_{i=1}^n \Gamma(\nu_i)} \times$$

$$\times \int_{\alpha_i} \delta\left(1 - \sum \alpha_i\right) \frac{u^{n-(L+1)d/2}}{(-F)^{n-dL/2}} \tag{4.2.31}$$

where u and F are the Symanzik polynomials. More details on Weinzierl *Feynman Integrals*.

Let's see an **example**. The *sunrise/sunset* integral (massless propagator):



where we have:

$$M = \begin{pmatrix} x_1 + x_2 & x_3 \\ x_3 & x_2 + x_3 \end{pmatrix} \quad ; \quad \vec{Q}^{\,\mu} = \begin{pmatrix} -x_3 p^\mu & -x_3 p^\mu \end{pmatrix} \tag{4.2.32}$$

$$J = x_3 p^2 \tag{4.2.33}$$

so:

$$u = x_1 x_2 + x_1 x_3 + x_2 x_3 \tag{4.2.34}$$

$$F = p^2 x_1 x_2 x_3 \tag{4.2.35}$$

that implies:

$$I^{(2)[4-2\epsilon]}_{3,\text{sunrise}} = \frac{\Gamma(-1+2\epsilon)}{(4\pi)^{4-2\epsilon}} \int \prod_{i=1}^{3} \mathrm{d}x_i \, \delta(1 - x_1 - x_2 - x_3)(-p^2 x_1 x_2 x_3)^{1-2\epsilon} \times \\ \times (x_1 x_2 + x_2 x_3 + x_1 x_3)^{-3(1-\epsilon)} \tag{4.2.36}$$

where, if we want, we can use:

$$\Gamma(-1+2\epsilon) = \frac{-\Gamma(1+2\epsilon)}{2\epsilon(1-2\epsilon)} \tag{4.2.37}$$

Direct integration in Feynman parameter space requires a bit of thought. The $1/\epsilon$ pole is quite easy since we may expand the integrand around $\epsilon = 0$. The kinematic dependence can also be factored out:

$$I^{(2)[4-2\epsilon]}_{3,\text{sunrise}} = \left(\frac{C_\Gamma}{(4\pi)^{2-\epsilon}} \right)^2 (-p^2)^{1-2\epsilon} \hat{I}^{(2)[4-2\epsilon]}_{3,\text{sunrise}} \tag{4.2.38}$$

where we have to remember:

$$C_\Gamma = \frac{\Gamma(1-\epsilon)^2 \Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \tag{4.2.39}$$

and the normalized integral is given by:

$$\hat{I}^{(2)[4-2\epsilon]}_{3,\text{sunrise}} = -\frac{\Gamma(1+2\epsilon)}{2\epsilon(1-2\epsilon)C_\Gamma^2}\int_{x_i}\delta(1-\sum x_i)(x_1x_2x_3)^{1-2\epsilon}\times$$
$$\times\,(x_1x_2+x_1x_3+x_2x_3)^{-3(1-\epsilon)} \quad (4.2.40)$$
$$= -\frac{1}{4\epsilon}-\frac{13}{8}-\frac{115}{16}\epsilon-\left(\frac{865}{32}-\frac{3}{2}\zeta_3\right)\epsilon^2-$$
$$-\left(\frac{5971}{64}-\frac{39}{4}\zeta_3-\frac{\pi^4}{40}\right)\epsilon^3+\mathcal{O}(\epsilon^4). \quad (4.2.41)$$

One quick method to check this is numerical integration with many digits plus fittin to an ansatz of trascendental numbers (PSLQ algorithm).

4.3 Passarino-Veltman tensor reduction

We have already seen how to deal with higher rank numerators in Feynman or Schwinger parameter space. There is a more direct method to perform the reduction in momentum space and to determine a basis of Feynman integral at 1-loop. The principles are to:

a. Identify a *basis of possible tensor structures* after integration using p_i^μ external momentum and $g^{\mu\nu}$ metric tensor.

b. Contract the tensor integrand (in d-dimension) with each element of the tensor basis to construct a linear system of equations for the *form factors*.

Example: rank 1 bubble. We have:

$$I_2^{(1)[d]}[k^\mu] = \int_k \frac{k^\mu}{D(k,m)D(k-p,m)} \quad (4.3.1)$$
$$= b_1 p^\mu \quad (4.3.2)$$

where we applied the step a indicating b_1 as form factor and p^μ the tensor basis. Note that we use $[k^\mu]$ as a slightly more general notation indicating the numerator.

If we apply the step b:

$$I_2^{(1)[d]}[k \cdot p] = b_1 p^2 \tag{4.3.3}$$
$$= \int_k \frac{k \cdot p}{D(k,m)D(k-p,m)} \tag{4.3.4}$$
$$= \int_k \frac{1}{2} \frac{D(k,m) - D(k-p,m) + p^2}{D(k,m)D(k-p,m)} \tag{4.3.5}$$
$$= \frac{1}{2}\Big(\int_k \frac{1}{D(k-p,m)} - \int_k \frac{1}{D(k,m)} + + p^2 \int_k \frac{1}{D(k,m)D(k-p,m)} \Big) \tag{4.3.6}$$

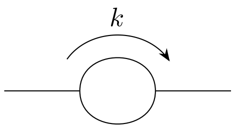
where the first and second term cancel out after a shift integration $k \to k+p$ in the first one. Finishing the calculation:

$$I_2^{(1)[d]}[k \cdot p] = \frac{1}{2} p^2 I_2^{(1)[d]}[1] \tag{4.3.7}$$

so we find:

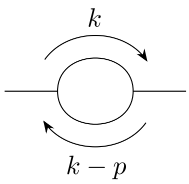
$$b_1 = \frac{1}{2} I_2^{(1)[d]}[1]. \tag{4.3.8}$$

We can also write these steps graphically:

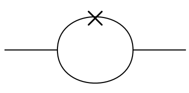
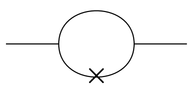


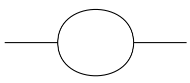
$[k^\mu] = I_2^{(1)[d]}[k^\mu]$ (4.3.9)

then:

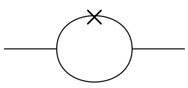
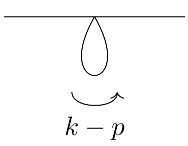


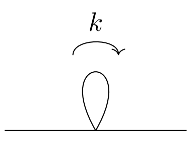
$[k \cdot p] = b_1 p^2$ (4.3.10)

$= \frac{1}{2}$  $[1] - \frac{1}{2}$  $[1]$

$+ \frac{p^2}{2}$  $[1]$ (4.3.11)

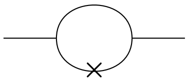
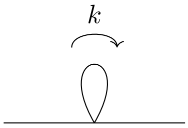
where indicates a cancelled propagator. We also have:

 $[1] =$ 

$[1] =$ 

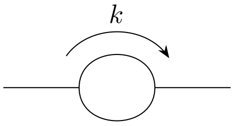
$[1]$ (4.3.12)

where to write the last loop we shifted the loop momentum. We have also:

 $[1] =$ 

$[1]$ (4.3.13)

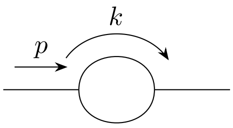
so we obtain cancelling the other two contribution:

$b_1 = \frac{1}{2}$ 

$[1].$ (4.3.14)

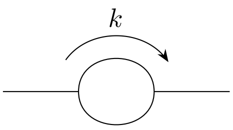
The principle is straightforward to apply to higher rank and higher multiplicity integrals though the expressions can become quite lengthly.

Example: rank 2 tensor bubble. We write the tensor basis:

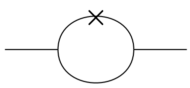
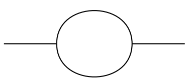


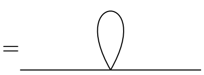
$[k^\mu k^\nu] = b_{00} g^{\mu\nu} + b_{11} p^\mu p^\nu$ (4.3.15)

so we need to determine two form factor, and now contract with:

$g_{\mu\nu}:$ 

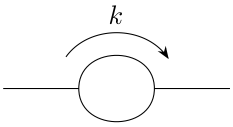
$[k^2] = d b_{00} + b_{11} p^2$ (4.3.16)

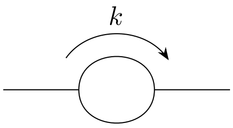
$=$  $[1] + m^2$  $[1]$

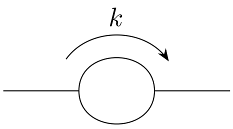
$=$  $[1]$

(4.3.17)

and the second element of the basis:

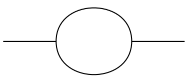
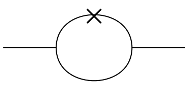
$p_\mu p_\nu$:  $[(k \cdot p)^2] = b_{00}p^2 + b_{11}p^4$ (4.3.18)

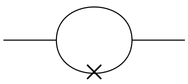
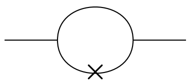
$=$  $[\left(\frac{1}{2}(D(k,m) - D(k-p,m)) + p^2\right)^2]$ (4.3.19)

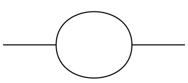
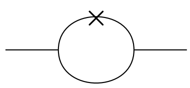
$= \frac{1}{4}$  $[D(k,m)^2 + D(k-p,m)^2 -$

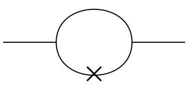
$- 2D(k,m)D(k-p,m) + p^4 + 2p^2(D(k,m) - D(k-p,m))]$ (4.3.20)

where if we shift k to keep on second term then cancel the last bracket, so:

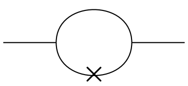
$= \frac{1}{4}\Big(p^4$  $[1] +$  $[D(k,m)] +$

$+$  $[D(k-p,m)] -$  $[D(k,m)]\Big)$ (4.3.21)

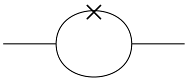
$= \frac{1}{4}\Big(p^2$  $[1] +$  $[D(k,m)] -$

$- 2$  $[D(k,m)]\Big).$ (4.3.22)

We define:

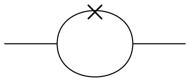
 $[D(k,m)] = \int_k \frac{k^2 - m^2}{k^2 - m^2} \equiv 0$ (4.3.23)

so we can perform:

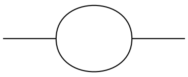
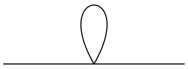
 $$[D(k,m)] = \int_k \frac{k^2 - m^2}{(k-p)^2 - m^2} \quad (4.3.24)$$

$$= \int_k \frac{(k+p)^2 - m^2}{k^2 - m^2} \quad (4.3.25)$$

$$= \int_k \frac{k^2 - m^2}{k^2 - m^2} + 2\int_k \frac{k \cdot p}{k^2 - m^2} + p^2 \int_k \frac{1}{k^2 - m^2} \quad (4.3.26)$$

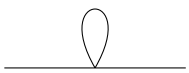
$= p^2$  $[1]$ (4.3.27)

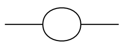
that implies:

$\frac{1}{4}p^4$  $[1] + \frac{1}{2}p^2$  $[1] = p^2 b_{00} + p^4 b_{11}.$ (4.3.28)

So we have determined a system of equations for b_{00} and b_{11}:

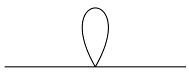
$$\begin{pmatrix} d & p^2 \\ 1 & p^2 \end{pmatrix} \begin{pmatrix} b_{00} \\ b_{11} \end{pmatrix} = \begin{pmatrix} 1 & m^2 \\ 1/2 & 1/4p^2 \end{pmatrix} \begin{pmatrix} \text{(tadpole)}\,[1] \\ \text{(bubble)}\,[1] \end{pmatrix} \quad (4.3.29)$$

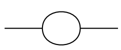
 [1]

 [1]

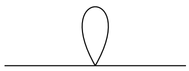
so to determine the coefficient we need to invert the matrix:

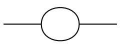
$$\begin{pmatrix} b_{00} \\ b_{11} \end{pmatrix} = \frac{1}{p^2(d-1)} \begin{pmatrix} p^2 & -p^2 \\ -1 & d \end{pmatrix} \begin{pmatrix} m^2 \\ \frac{1}{2}\frac{1}{4}p^2 \end{pmatrix} \begin{pmatrix} \text{(tadpole)}\,[1] \\ \text{(bubble)}\,[1] \end{pmatrix} \quad (4.3.30)$$

 [1]

 [1]

$$= \frac{1}{2(d-1)} \begin{pmatrix} 1 & -\frac{p^2}{2}\left(1 - \frac{4m^2}{p^2}\right) \\ -\frac{(d-2)}{p^2} & \frac{1}{2}\left(d - \frac{4m^2}{p^2}\right) \end{pmatrix} \begin{pmatrix} \text{(tadpole)}\,[1] \\ \text{(bubble)}\,[1] \end{pmatrix}. \quad (4.3.31)$$

 [1]

 [1]

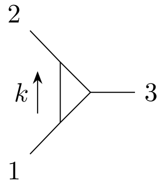
Worked example Perform the tensor reduction for rank 1,2 and 3 triangle integral in the massless limit with kinematics:

$$p_1^2 = p_2^2 = 0 \quad ; \quad p_3^2 \neq 0. \quad (4.3.32)$$

We define:

$$I_3^{(1)[d]}[\mathcal{N}] = \int_k \frac{\mathcal{N}}{D(k,0)D(k+p_2)D(k-p_1)} \tag{4.3.33}$$

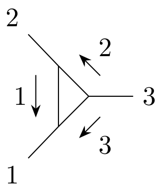
$$= \tag{4.3.34}$$



where we have:

$$\mathcal{N} = \begin{pmatrix} k^{\mu_1} & k^{\mu_1}k^{\mu_2} & k^{\mu_1}k^{\mu_2}k^{\mu_3} \end{pmatrix}. \tag{4.3.35}$$

The solution is:



with:

$$D = \left\{ k^2 \ , \ (k+p)^2 \ , \ (k-p_1)^2 \right\} \tag{4.3.36}$$

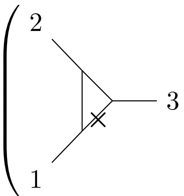
at *rank 1* we have the tensor basis $\{p_1^\mu, p_2^\mu\}$ with form factors $\{c_1, c_2\}$, so we form the system of equation form:

$$k \cdot p_1 = \frac{1}{2}\left(k^2 - (k-p_1)^2 + p_1^2\right) \tag{4.3.37}$$

$$k \cdot p_2 = \frac{1}{2}\left((k+p_2)^2 - k^2 - p_2^2\right) \tag{4.3.38}$$

in addition take $p_i^2 = 0$:

$\frac{1}{2}\Big($



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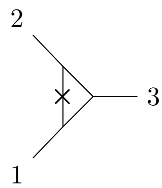
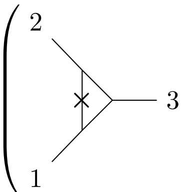


$\Big) = c_2\, p_1 \cdot p_2 \qquad (4.3.39)$

$\frac{1}{2}\Big($



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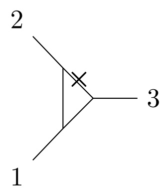
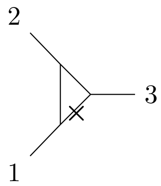
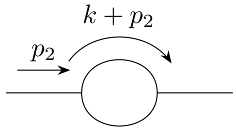


$\Big) = c_1\, p_1 \cdot p_2. \qquad (4.3.40)$

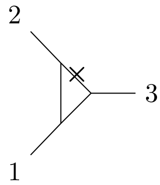
Let us look at this integral to start:



$=$


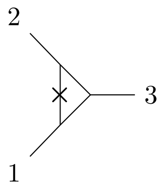
$$\underset{p_2^2=0}{=} 0 \tag{4.3.41}$$

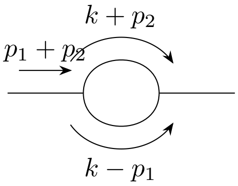
so a scaleless integral, but we also have for symmetry:



$$= 0 \tag{4.3.42}$$

so another scaleless integral. We can see:



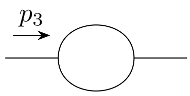
$=$


$$= I_2^{(1)[d]}((p_1 + p_2)^2) \tag{4.3.43}$$

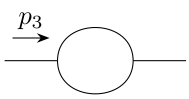
where:

$$(p_1 + p_2)^2 = 2p_1 \cdot p_2 = p_3^2 = s_3 \tag{4.3.44}$$

which is invariant. In this way we get:

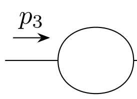
$-\frac{1}{2}$


$$= \frac{1}{2} s_3 c_2 \tag{4.3.45}$$

$\frac{1}{2}$


$$= \frac{1}{2} s_3 c_1 \tag{4.3.46}$$

so we can write:

$$\{c_1, c_2\} = \frac{1}{s_3}\{1, -1\}$$


$$. \tag{4.3.47}$$

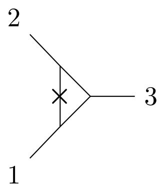
At *rank 2* we have the tensor basis:

$$\left\{ g^{\mu_1\mu_2} \ , \ p_1^{\mu_1} p_1^{\mu_2} \ , \ p_2^{\mu_1} p_2^{\mu_2} \ , \ \frac{1}{2}(p_1^{\mu_1} p_2^{\mu_2} + p_1^{\mu_2} p_2^{\mu_1}) \right\} \tag{4.3.48}$$

with form factors:

$$\{c_{00} \ , \ c_{11} \ , \ c_{22} \ , \ c_{12}\}. \tag{4.3.49}$$

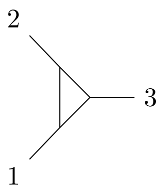
Contracting with $g_{\mu_1\mu_2}$ is quite easy:



$$= dc_{00} + p_1 \cdot p_2 c_{12} \tag{4.3.50}$$

$$= dc_{00} + \frac{1}{2} s_3 c_{12} \tag{4.3.51}$$

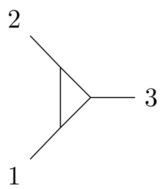
contracting with $p_{1\mu_1} p_{1\mu_2}$ gives:



$$[(k \cdot p_1)^2] = (p_1 \cdot p_2)^2 c_{22} \tag{4.3.52}$$

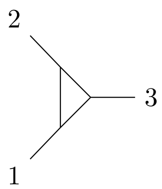
$$= \frac{1}{4} s_2^3 c_{22} \tag{4.3.53}$$

$$=$$



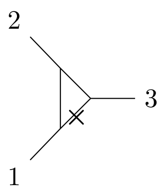
$$\left[\frac{1}{4}(D_3 - D_1)^2\right] \tag{4.3.54}$$

$$=$$

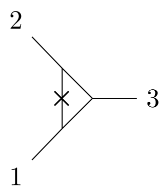


$$\left[\frac{1}{4}(D_3^2 + D_1^2 - 2D_1 D_3)\right] \tag{4.3.55}$$

$$=$$

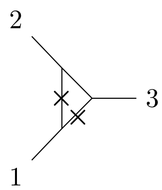


$$\left[\frac{1}{4}(k + p_1)^2\right] +$$



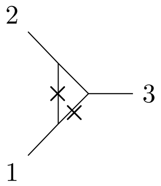
$$\left[\frac{1}{4} k^2\right] -$$

$$-\frac{1}{2}$$



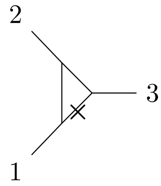
$$[1] \tag{4.3.56}$$

but the last graph is scaleless:

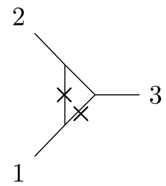


$$= \int_k \frac{1}{D_2} = \int_k \frac{1}{(k-p_2)^2} = \int_k \frac{1}{k^2} = 0. \tag{4.3.57}$$

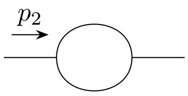
We have:



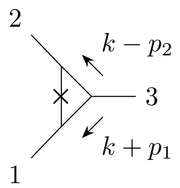
$$\left[\frac{1}{4}k^2 + \frac{1}{2}k\cdot p_1 + \underbrace{\frac{1}{4}p_1^2}_{} = 0\right]$$



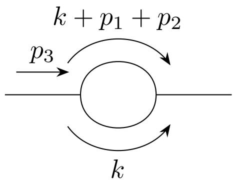
$$= \quad [\text{triangle}] \quad + \frac{1}{2}p_1^\mu \quad [\text{bubble}] \quad [k_\mu] = 0 \tag{4.3.58}$$



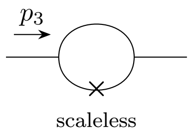
because both of the diagrams are scaleless. We can also see:

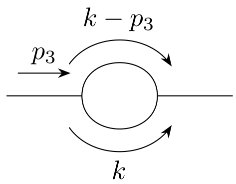


$$\left[\frac{1}{4}k^2\right] = \quad [\text{bubble}] \quad \left[\frac{1}{4}(k+p_2)^2\right] \tag{4.3.59}$$

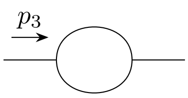


$$= \frac{1}{4} \quad [\text{bubble}] \quad + \frac{1}{2}p_2^\mu \quad [\text{bubble}] \quad [k_\mu] \tag{4.3.60}$$





$$= \frac{1}{4}p_2\cdot p_3 \quad [\text{bubble}] \quad [1] \tag{4.3.61}$$

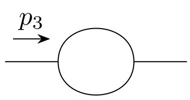


where we saw a rank 1 bubble, ad we have:

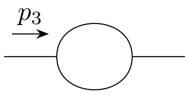
$$2p_2\cdot p_3 = (p_2+p_3)^2 - s_3 = \underset{=0}{p_1^2} - s_3 \tag{4.3.62}$$

so we get:

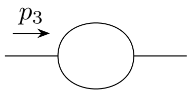
$$\frac{s_3}{8} \quad [\text{bubble}] \quad = \frac{1}{4}s_3^2 c_{22}. \tag{4.3.63}$$



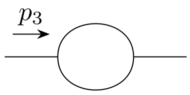
The contraction with $p_{2\mu_1}p_{2\mu_2}$ is similar and it gives us:

$$\frac{s_3}{8}$$  $$= \frac{1}{4}s_3^2 c_{11}. \quad (4.3.64)$$

The contraction with $\frac{1}{2}(p_{1\mu_1}p_{2\mu_2} + p_{1\mu_2}p_{2\mu_3})$ gives us:

$$\frac{s_3}{8}$$  $$= \frac{1}{2}s_3^2 c_{00} + \frac{1}{8}s_3^2 c_{12}. \quad (4.3.65)$$

We can see all the equation ad a matrix equation.

$$\mathcal{M} \cdot \vec{c} = \vec{b}$$  (4.3.66)

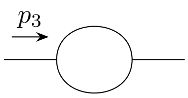
where:

$$\vec{c} = \begin{pmatrix} c_{00} \\ c_{11} \\ c_{22} \\ c_{33} \end{pmatrix} \qquad ; \qquad \vec{b} = \begin{pmatrix} 1 \\ s_3/8 \\ s_3/8 \\ s_3/8 \end{pmatrix} \tag{4.3.67}$$

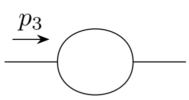
and the matrix:

$$\mathcal{M} = \begin{pmatrix} d & 0 & 0 & s_3/2 \\ 0 & 0 & s_3^2/4 & 0 \\ 0 & s_3^2/4 & 0 & 0 \\ s_3/2 & 0 & 0 & s_3^2/8 \end{pmatrix}. \tag{4.3.68}$$

We can see we get:

$$\vec{c} = \begin{pmatrix} \frac{1}{2(d-2)} \\ -\frac{1}{2s_3} \\ -\frac{1}{2s_3} \\ \frac{d-4}{(d-2)s_3} \end{pmatrix} \times$$  (4.3.69)

where we can remember:

 $$= I_2^{(1)[d]}(s_3). \quad (4.3.70)$$

At *rank 3* the algebra is better implemented on a computer. the final resultr however is relatively simple though the rank 2 bubble reduction formula is required at intermediate stages. the tensor basis is (6 elements):

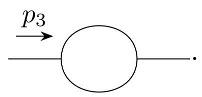
$$\vec{T} \equiv \{g^{\mu_1\mu_2}p_1^{\mu_3} + \text{cyclic}\ , g^{\mu_1\mu_2}p_2^{\mu_3} + \text{cyclic}\ ,$$
$$p_1^{\mu_1}p_1^{\mu_2}p_1^{\mu_3}\ ,\ p_2^{\mu_1}p_2^{\mu_2}p_2^{\mu_3}\ ,\ p_1^{\mu_1}p_1^{\mu_2}p_2^{\mu_3} + \text{cyclic}\ ,\ p_1^{\mu_1}p_2^{\mu_2}p_2^{\mu_3} + \text{cyclic}\} \tag{4.3.71}$$

in the only 1 basis integral as before, the form factors vector is:

$$\vec{c} = \{c_{001}\ ,\ c_{002}\ ,\ c_{111}\ ,\ c_{222}\ ,\ c_{112}\ ,\ c_{122}\}. \tag{4.3.72}$$

Solving the system results in:

$$\vec{c} = \frac{1}{4(d-1)}\left\{-1\ ,\ 1\ ,\ \frac{d}{s_3}\ ,\ \frac{-d}{s_3}\ ,\ \frac{-(d-4)}{s_3}\ ,\ \frac{d-4}{s_3}\right\} \times$$



(4.3.73)

4.4 Clifford algebra in d-dimension

There are some subleties in the extension of the Clifford algebra to $d = 4-2\epsilon$ dimension and the represantation of the γ matrices.

Since we will deal with QED and QCD which are parity symmetric, the issue will not arise, but for the EW part of the Standard Model we must pay attention.

The Clifford algebra we have used so far, in 4 dimension, is:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1} \tag{4.4.1}$$

where γ^μ are 4x4 matrices. If we extend our metric to live in d dimensions we may derive identities such as:

$$\gamma^\mu\gamma_\mu = \frac{1}{2}\{\gamma^\mu, \gamma_\mu\} \tag{4.4.2}$$
$$= g^\mu_\mu\mathbb{1} \tag{4.4.3}$$
$$= d\mathbb{1} \tag{4.4.4}$$
$$\gamma^\mu\gamma^\nu\gamma_\mu = -\gamma^\mu\gamma_\mu\gamma^\nu + 2\gamma^\nu \tag{4.4.5}$$
$$= (2-d)\gamma^\nu \tag{4.4.6}$$

which will have to be applied to the numerators of the Feynman diagrams. We don't need an explicit represantation of γ^μ in order to do this, but in principle we should worry about it.

It is possible to construct explicit representations for γ^μ is arbitrary dimensions, however the role of γ_5 in defining chirality only make sense in $4d$. But, what is γ_5 in $d = 4-2\epsilon$?

SO(2) γ **matrices** We have $\gamma^i_{(2)}$ that are 2x2 matrices where $i = 1, 2$. They satisfy:

$$\left(\gamma^i_{(2)}\right)^\dagger = \gamma^i_{(2)} \tag{4.4.7}$$

and we may choose:

$$\gamma^1_{(2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (= \sigma_1) \tag{4.4.8}$$

$$\gamma^2_{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} (= \sigma_2) \tag{4.4.9}$$

so that:

$$\left\{\gamma^i_{(2)}, \gamma^j_{(2)}\right\} = 2\delta^{ij}\mathbb{1}_{2x2}. \tag{4.4.10}$$

In even dimension we may define a *chirality* operation:

$$\hat{\gamma}^3_{(2)} = \frac{i}{2}\left(\gamma^1_{(2)}\gamma^2_{(2)} - \gamma^2_{(2)}\gamma^1_{(2)}\right) \tag{4.4.11}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.4.12}$$

$SO(3)$ γ **matrices** The dimension of the γ represantation is $2^{\lfloor d/2 \rfloor}$ for integer dimension d where $\lfloor \cdot \rfloor$ is the *floor* operation which round down to the nearest integer. So in $3d$ we have 3, 2x2 matrices satisfying:

$$\left\{\gamma^i_{(3)}, \gamma^j_{(3)}\right\} = 2\delta^{ij}\mathbb{1}_{2x2} \tag{4.4.13}$$

where:

$$\gamma^1_{(3)} = \gamma^1_{(2)} \tag{4.4.14}$$

$$\gamma^2_{(3)} = \gamma^2_{(2)} \tag{4.4.15}$$

$$\gamma^3_{(3)} = \hat{\gamma}^3_{(2)}. \tag{4.4.16}$$

However there is no notion of chirality in $3d$. Higher dimension represantation can be built from tensor products, i.e. $d = 4$ 4x4 matrices built form tensor product of 2x2 matrices.

$SO(1,1)$ **matrices** In $2d$ Minkowski space the γ matrices satisfy (with the usual $g^{\mu\nu} = \text{diag}(1, -1)$):

$$(\gamma^\mu_{(2)})^\dagger = \begin{cases} \gamma^\mu_{(2)} & \mu = 0 \\ -\gamma^\mu_{(2)} & \mu = 1 \end{cases} \tag{4.4.17}$$

or:

$$(\gamma^\mu_{(2)})^\dagger = \gamma_{(2)\mu}. \tag{4.4.18}$$

Choosing:

$$\gamma^0_{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (= \sigma_3) \tag{4.4.19}$$

$$\gamma^1_{(2)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (= \sigma_1). \tag{4.4.20}$$

We find explicitly:

$$\left\{\gamma^{\mu}_{(2)}, \gamma^{\nu}_{(2)}\right\} = 2g^{\mu\nu}\mathbb{1}_{2x2}. \tag{4.4.21}$$

We may also define a *chirality* operation:

$$\hat{\gamma}^{3}_{(2)} = \frac{1}{2}\left(\gamma^{0}_{(2)}\gamma^{1}_{(2)} - \gamma^{1}_{(2)}\gamma^{0}_{(2)}\right) \tag{4.4.22}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.4.23}$$

The operator anti-commutes with $\gamma^{\mu}_{(2)}$:

$$\left\{\gamma^{\mu}_{(2)}, \hat{\gamma}^{3}_{(2)}\right\} = 0. \tag{4.4.24}$$

The $3d$ case is constructed analagously with the $SO(3)$ example.

$SO(1,3)$ **matrices** The Standard Dirac matrices in the Weyl basis can be built from the $SO(1,1)$ matrices:

$$\gamma^{0}_{(4)} = \hat{\gamma}^{3}_{(2)} \otimes \mathbb{1}_{2x2} \tag{4.4.25}$$

$$\gamma^{1}_{(4)} = \gamma^{2}_{(2)} \otimes \hat{\gamma}^{3}_{(2)} \tag{4.4.26}$$

$$\gamma^{2}_{(4)} = -i\gamma^{2}_{(2)} \otimes \gamma^{2}_{(2)} \tag{4.4.27}$$

$$\gamma^{3}_{(4)} = \gamma^{2}_{(2)} \otimes \gamma^{1}_{(2)} \tag{4.4.28}$$

in this way:

$$\hat{\gamma}^{5}_{(4)} = -i\gamma^{0}_{(4)}\gamma^{1}_{(4)}\gamma^{2}_{(4)}\gamma^{3}_{(4)} \tag{4.4.29}$$

and:

$$P_{\pm} = \frac{1 \pm \hat{\gamma}^{5}_{(4)}}{2}. \tag{4.4.30}$$

4.4.1 γ matrices in dimensional regularisation

If we were to think of representing the γ in $d = 4 - 2\epsilon$ dimension as an ∞ dimension vector space we would not find a correct continuation around $d = 4$. We need γ_5 to represent *chirality* in $d = 4$.

Therefore we keep the dimension of the spinor space to be 4:

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}\mathbb{1}_{4x4} \tag{4.4.31}$$

where we have $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, \dots)$. We therefore keep the identities for $\gamma^{\mu}\gamma_{\mu}$ etc. as before.

Chirality is defined in $4{-}d$ hence:

$$\hat{\gamma}^{5} = i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3} \tag{4.4.32}$$

such that:

$$\left\{\gamma^\mu, \hat{\gamma}^5\right\} = 0 \qquad , \quad \mu = (0, 1, 2, 3) \tag{4.4.33}$$

but for $\mu > 3$ we must implement a commutation relation:

$$\left[\gamma^\mu, \hat{\gamma}^5\right] = 0 \qquad , \quad \mu > 3. \tag{4.4.34}$$

This is the *'t Hooft-Veltman* scheme.

Chapter 5

Renormalization of QED at one-loop

In this chapter, as the title says, we will see how to renormalize the QED. In particular we complete renormalization of QED at one-loop though renormalization of A_μ, ψ, e and m. We will see the quantum corrections to EM interaction and we will see $g-2=\alpha/(2\pi)$, so the *anomalous magnetic moment*. Also, we'll see the quantum corrections to electric potential (Uehling potential) and the charge screening. Last but not least we will see how gauge invariance, $Z_1 = Z_2$ and the Ward-Takahashi identity are related each orther.

5.1 Lagrangian, Feynman rules and counter terms

Let's see a brief rewiew of some things we already know before jump into the renormalization of QED. We consider a general R_ξ *covariant gauge*.

5.1.1 Bare Lagrangian and gauge invariance

We know the lagrangian of our theory:

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{\mathcal{D}} - m)\psi + \mathcal{L}_{gf} \tag{5.1.1}$$

where we have:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{5.1.2}$$

$$\mathcal{D}^\mu = \partial^\mu + ieA^\mu \tag{5.1.3}$$

and we know is invariant under local $U(1)$ rotation:

$$\psi \longrightarrow \psi' = e^{i\theta(x)}\psi \tag{5.1.4}$$

$$A_\mu \longrightarrow A'_\mu = A_\mu - \frac{1}{e}\partial_\mu. \tag{5.1.5}$$

The covariant gauge fixing term is:

$$\mathcal{L}_{gf} = \frac{1}{2\xi}(\partial_\mu A^\mu)^2. \tag{5.1.6}$$

To derive the form of the photon propagator we take the terms in $\mathcal{L}_{QED}$ with 2 photon fields A^μ:

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{\xi}(\partial_\mu A^\mu)^2 = -\frac{1}{2}\left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu + \frac{1}{\xi}\partial_\mu A^\mu \partial_\nu A^\nu\right) \tag{5.1.7}$$

$$= \frac{1}{2}A^\mu \left(\Box g_{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial_\mu \partial_\nu\right) A^\nu + \text{surface term} \tag{5.1.8}$$

where we have used integration by parts and removed surface terms from the $\mathcal{L}$. The Fourier tranformation of (5.1.8) gives us:

$$\frac{1}{2}\tilde{A}^\mu \mathcal{M}_{\mu\nu}\tilde{A}^\nu \tag{5.1.9}$$

where:

$$\mathcal{M}_{\mu\nu} = -p^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) p_\mu p_\nu. \tag{5.1.10}$$

The photon propagator $D_{\mu\nu}$ is defined by:

$$D_{\mu\nu}(p^2, \xi)\mathcal{M}^\nu{}_\sigma(p^2, \xi) = i g_{\mu\sigma} \tag{5.1.11}$$

so, as the inverse operator of $\mathcal{M}$. We solve (5.1.11) for $D_{\mu\nu}$ by first providing a tensor decomposition:

$$D_{\mu\nu}(p^2, \xi) = \frac{\alpha}{p^2}\left(-g_{\mu\nu} + \beta \frac{p_\mu p_\nu}{p^2}\right) \tag{5.1.12}$$

which can be inserted into (5.1.11) to find:

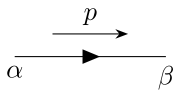
$$\alpha = i \qquad , \qquad \beta = 1 - \xi. \tag{5.1.13}$$

We need to add contour deformation to avoid the pole on the real axis, so the final expression is:

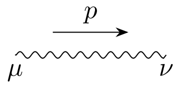
$$D_{\mu\nu}(p^2, \xi) = \frac{1}{p^2 + i\emptyset^+}\left(-g_{\mu\nu} + (1 - \xi)\frac{p_\mu p_\nu}{p^2}\right). \tag{5.1.14}$$

5.1.2 Feynman rules

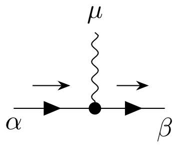
In this section we see the Feynman rules of the theory. We have the progators:



$$= \frac{i}{p^2 - m^2 + i\emptyset^+}(\not{p} + m)_{\beta\alpha} \tag{5.1.15}$$



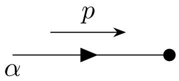
$$= \frac{i}{p^2 + i\emptyset^+}\left(-g_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2}\right) \tag{5.1.16}$$



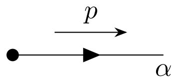
$$= -ie\gamma^\mu_{\alpha\beta} \tag{5.1.17}$$

plus the rules for external spinors and polarisations:

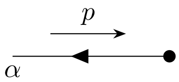
initial ; final (5.1.18)



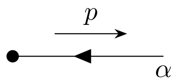
$= u_{\alpha,s}(p,m)$;



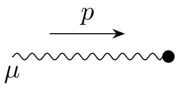
$= \overline{u}_{\alpha,s}(p,m)$ (5.1.19)



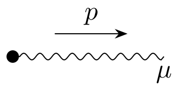
$= \overline{v}_{\alpha,s}(p,m)$;



$= v_{\alpha,s}(p,m)$ (5.1.20)

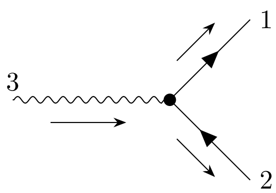


$= \epsilon_\mu(p)$;



$= \epsilon^*_\mu(p).$ (5.1.21)

For example we have the diagram:



$$= -ie\overline{u}_{s_1}(p_1,m)\not{\epsilon}(p_3)v_{s_2}(p_2,m). \tag{5.1.22}$$

5.1.3 Superficial degree of divergence

We know that the expression of the superficial degree of divergence is theory dependent, so recalling how we derived (3.1.6), where d was the dimension, L the loop order and n the number of internal lines; we may derive a similar form for a graph in QED:

$$\omega(\Gamma_{QED}) = dL - 2P_\gamma - P_e \tag{5.1.23}$$

where P_γ and P_e are the number of internal photons and electrons/positrons respectively.

We may, in terms of external photon and electron multiplicity, E_γ and E_e, using the expressions:

$$L = P_e + P_\gamma - V + 1 \tag{5.1.24}$$

and:

$$V = 2P_\gamma + E_\gamma \tag{5.1.25}$$
$$= P_e + \frac{1}{2}E_e \tag{5.1.26}$$

just to write:

$$\omega(\Gamma_{QED}) = d + \left(\frac{1-d}{2}\right)E_e + \left(\frac{2-d}{2}\right)E_\gamma + \left(\frac{d-4}{2}\right)V \tag{5.1.27}$$

so in $4d$ is simply:

$$\omega(\Gamma_{QED}) = 4 - \frac{3}{2}E_e - E_\gamma. \tag{5.1.28}$$

Using the expression (5.1.28) we can see the divergent graph in the table 5.1.

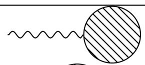
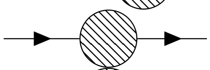
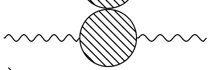
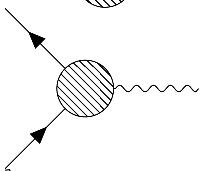
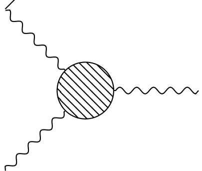
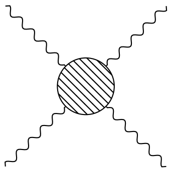
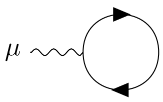
Γ	E_e	E_γ	$\omega(\Gamma_{QED})$	name
	0	1	3	tadpole
	2	0	1	electron self energy
	0	2	2	photon vacuum polarisation
	2	1	0	vertex
	0	3	1	-
	0	4	0	-

Table 5.1: Divergent diagrams of QED.

The treatment of the tadpole graph is different, and more straightforward, than in ϕ^3 where it was necessary to re-define the vacuum expectation value. There we can show the graph is zero e.g. at 1-loop:

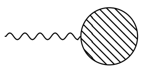


$$= -ie\,\mathrm{Tr}\left\{\gamma^\mu \int_k \frac{\not{k}+m}{k^2-m^2}\right\} \tag{5.1.29}$$

$$= -ie\,\underbrace{\mathrm{Tr}\{\gamma^\mu\}}_{=0}\, m \int_k \frac{1}{k^2-m^2} - ie\,\mathrm{Tr}\{\gamma^\mu\gamma^\nu\} \underbrace{\int_k \frac{k_\nu}{k^2-m^2}}_{=0} \tag{5.1.30}$$

$$= 0. \tag{5.1.31}$$

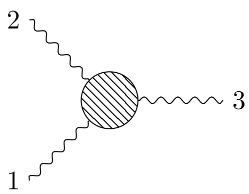
This property is valid to all loop orders, so:

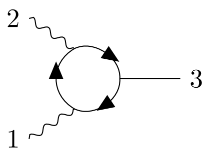


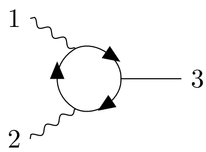
$$= 0. \tag{5.1.32}$$

There are two other graph to worry about it - the 3 and 4 γ graphs. There are operators for γ self-interactions in the lagrangian, so it's not clear where to absorb the divergences.

The resolution is to see that divergences cancel between contributing diagrams e.g.:



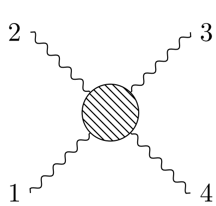
$=$


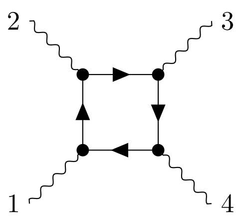
$+$


$+\ \mathcal{O}(e^5)$ (5.1.33)

$$= 0 \tag{5.1.34}$$

where with $\mathcal{O}(e^5)$ we indicate the higher loops. For the 4γ case we have:



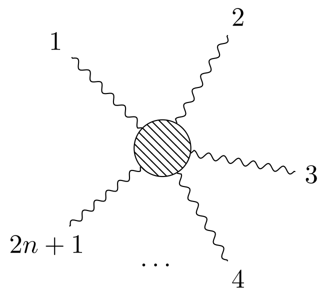
$=$


$+$ permutations (6 diagrams in total) $+\ \mathcal{O}(e^6)$ (5.1.35)

$$= \text{finite.} \tag{5.1.36}$$

The 3γ amplitude result is a special case of a more general theorem:

Teorema 1 (Furry's theorem) *Correlation functions and amplitudes for odd numbers of photons are zero:*



$$= \ 0 \quad , \qquad n \in \mathbb{Z}^+. \tag{5.1.37}$$

This theorem follows using *charge conjugation symmetry.*

5.1.4 Renormalization constants and counterterms

In terms of bare fields, couplings and masses we have:

$$\mathcal{L}_{QED} = \frac{1}{2} A_0^\mu \left(g_{\mu\nu}\Box - \left(1 - \frac{1}{\xi_0}\right)\partial_\mu\partial_\nu \right) A_0^\nu + \\ + i\overline{\psi}_0 \partial\!\!\!/ \psi_0 - m_0 \overline{\psi}_0 \psi_0 - e_0 \overline{\psi}_0 A\!\!\!/_0 \psi_0. \tag{5.1.38}$$

We relate bare quantities to renormalized ones using:

$$\psi_0 = \sqrt{Z_\psi}\psi_R \tag{5.1.39}$$

$$A_0^\mu = \sqrt{Z_A} A_R^\mu \tag{5.1.40}$$

$$m_0 = Z_m m \tag{5.1.41}$$

$$e_0 = Z_e e_R \mu_R^\epsilon = \frac{Z_1}{Z_\psi \sqrt{Z_A}} e_R \mu_R^\epsilon \tag{5.1.42}$$

$$\xi_0 = \frac{Z_A}{Z_\xi} = \xi_R \tag{5.1.43}$$

so that:

$$\mathcal{L}_{QED} = \mathcal{L}_{QED,R} + \mathcal{L}_{CT} \tag{5.1.44}$$

where we have:

$$\begin{aligned} \mathcal{L}_{CT} = {} & \frac{1}{2}(Z_A - 1) A_R^\mu (\Box g_{\mu\nu} - \partial_\mu\partial_\nu) A_R^\nu + \\ & + \frac{1}{2} \underbrace{(Z_\xi - 1)}_{\xi_R} A_R^\mu \partial_\mu \partial_\nu A_R^\nu + \\ & + i(Z_\psi - 1)\overline{\psi}_R \partial\!\!\!/ \psi_R - (Z_m Z_\psi - 1) m_R \overline{\psi}_R \psi_R - \\ & - (Z_1 - 1) e_R \mu_R^\epsilon \overline{\psi}_R A\!\!\!/_R \psi_R \end{aligned} \tag{5.1.45}$$

and the term $\mathcal{L}_{QED,R}$ which is invariant under gauge rotations:

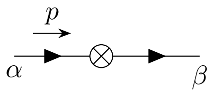
$$\psi_R \longrightarrow \psi'_R = e^{i\theta(x)}\psi_R \tag{5.1.46}$$

$$A^\mu_R \longrightarrow A'^\mu_R = A^\mu_R - \frac{1}{\mu^\epsilon_R e_R}\partial^\mu\theta(x). \tag{5.1.47}$$

For the whole expression to be invariant requires:

$$Z_1 = Z_\psi. \tag{5.1.48}$$

The Feynman rules for the counter-terms are:

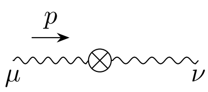


$$= i\left(\not{p}\delta_\psi - m\delta_3\right)_{\beta\alpha} \tag{5.1.49}$$

where:

$$\delta_3 = Z_m Z_\psi - 1 \quad , \quad \delta_\psi = Z_\psi - 1. \tag{5.1.50}$$

We also have:

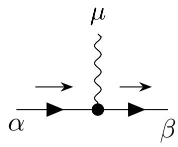


$$= -i\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right)\delta_A \tag{5.1.51}$$

where:

$$\delta_A = Z_A - 1. \tag{5.1.52}$$

The last rule is:



$$= -ie_R\mu^\epsilon_R\gamma^\mu\delta_1 \tag{5.1.53}$$

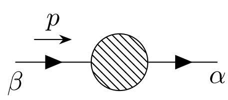
where:

$$\delta_1 = Z_1 - 1. \tag{5.1.54}$$

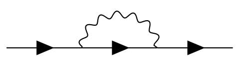
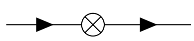
We will show later that $Z_\xi = 1$.

5.2 Divergent graph at 1-loop

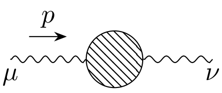
The expansions of the divergent graph to one-loop are the following. For the *electron self energy*:



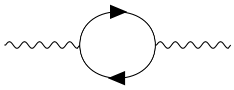
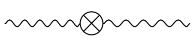
$$= \Sigma_{\beta\alpha}(p, m) \tag{5.2.1}$$

$$=$$  $$+$$  $$+\ \mathcal{O}(e^4). \tag{5.2.2}$$

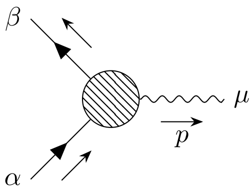
For the *vacuum polarisation* we have:



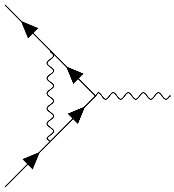
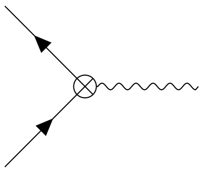
$$= \Pi_{\mu\nu}(p, m) \tag{5.2.3}$$

$$= $$  $$+$$  $$+\ \mathcal{O}(e^4). \tag{5.2.4}$$

The *vertex correction* is:

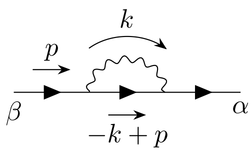


$$= \Gamma^{\mu}_{\beta\alpha}(p, m) \tag{5.2.5}$$

$$=$$  $$+$$  $$+\ \mathcal{O}(e^5). \tag{5.2.6}$$

5.2.1 Electron self-energy

Now we take, to start with, $\xi_R = 1$ (*Feynman gauge*). We have:



$$= (-ie_R\mu_R)^2 \int_k \gamma^{\mu}_{\alpha\delta} \frac{i(-\not{k} + \not{p} + m_R)_{\delta\sigma}}{D(k-p, m_R)} \gamma^{\nu}_{\sigma\beta} \left(\frac{-ig_{\mu\nu}}{D(k,0)} \right) \tag{5.2.7}$$

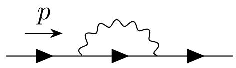
$$= e_R^2 \mu_R^{2\epsilon} \int_k \frac{(\gamma^\mu(\not{k} - \not{p} - m_R)\gamma_\mu)_{\alpha\beta}}{D(k,0)D(k-p, m_R)} \tag{5.2.8}$$

$$= e_R^2 \mu_R^{2\epsilon} \int_k \frac{((2-d)(\not{k} - \not{p}) - dm_R)_{\alpha\beta}}{D(k,0)D(k-p, m_R)} \tag{5.2.9}$$

and we can define:

$$I^{(1)[d]}_{2,m_1m_2}[\mathcal{N}] = \int_k \frac{\mathcal{N}}{D(k, m_1)D(k-p, m_2)} \tag{5.2.10}$$

so that:



$$= -e_R^2\mu_R^{2\epsilon}\Big\{\big[(2-d)\not{p} + dm_R\big]I_{2,m}^{(1)[d]} - (2-d)I_{2,m}^{(1)[d]}[\not{k}]\Big\} \tag{5.2.11}$$

$$= -e_R^2\mu_R^{2\epsilon}\Big[\frac{(2-d)\not{p}}{2p^2}\Big((p^2+2m_R^2)I_{2,m}^{(1)[d]}[1] - I_{1,m}^{(1)[d]}\Big) + + dm_R I_{2,m}^{(1)[d]}\Big] \tag{5.2.12}$$

where we have used the fact:

$$I_{2,m}^{(1)[d]}[\not{k}] = \not{p}b_1(p;0,m) \tag{5.2.13}$$

$$= \frac{\not{p}}{2p^2}\Big[I_{1,m}^{(1)[d]}[1] + (p^2 - m_R^2)I_{2,m}^{(1)[d]}[1]\Big]. \tag{5.2.14}$$

The scalar integrals $d = 4-2\epsilon$ dimensional are:

$$I_{1,m}^{(1)[4-2\epsilon]}[1] = \frac{i}{(4\pi)^{2-\epsilon}}\frac{\Gamma(1+\epsilon)}{(1-\epsilon)\epsilon}(m_R^2)^{1-\epsilon} \tag{5.2.15}$$

$$= \frac{iC_\Gamma}{(4\pi)^2}m_R^2\left(\frac{1}{\epsilon} + 1 - \log(m_R^2) + \mathcal{O}(\epsilon)\right) \tag{5.2.16}$$

and:

$$I_{2,0m}^{(1)[4-2\epsilon]}[1] = \frac{iC_\Gamma}{(4\pi)^2}\left(\frac{1}{\epsilon} + 2 - \log(m_R^2) + \left(1 - \frac{m_R^2}{p^2}\right)\log\left(1 - \frac{p^2}{m_R^2}\right) + \mathcal{O}(\epsilon)\right) \tag{5.2.17}$$

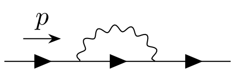
where:

$$C_\Gamma = \Gamma(1+\epsilon)(4\pi)^\epsilon \tag{5.2.18}$$

and we find:

$$e_R^2 = 4\pi\alpha. \tag{5.2.19}$$

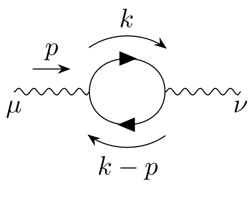
We can conclude:



$$= i\left(\frac{\alpha}{4\pi}\right)(\not{p} - 4m_R)\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0). \tag{5.2.20}$$

5.2.2 Photon vacuum polarisation

We can compute (with a minus sing for the closed fermion loop):

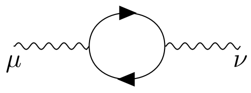


$$= -(-ie_R\mu_R)^2\int_k \frac{\mathrm{Tr}\big\{\gamma^\mu i(\not{k}+m_R)\gamma^\nu i(\not{k}-\not{p}+m_R)\big\}}{D(k,m_R)D(k-p,m_R)} \tag{5.2.21}$$

where we have:

$$\mathrm{Tr}\left\{\gamma^\mu(\not{k}+m_R)\gamma^\nu(\not{k}-\not{p}+m_R)\right\} = 4\Big(2k^\mu k^\nu - k^2 g^{\mu\nu} - k^\mu p^\nu - \\ - k^\nu p^\mu + k\cdot p g^{\mu\nu} + m_R^2 g^{\mu\nu}\Big) \quad (5.2.22)$$

so we need the tensor reduction for up to rank 2 bubble integrals. After applying Passarino-Veltman reduction we find:

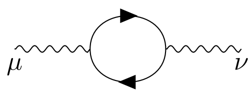


$$= -e_R^2\mu_R^{2\epsilon}\Big(-g^{\mu\nu}p^2 + p^\mu p^\nu\Big)A \quad (5.2.23)$$

with:

$$A = \frac{4}{p^2(d-1)}\left[(d-2)I_{1,m}^{(1)[d]} - \left(2m_R^2 + \frac{d-2}{2}p^2\right)I_{2,mm}^{(1)[d]}\right]. \quad (5.2.24)$$

We can see that the bubble integranl is the same we used in ϕ^3 theory, so in $d = 4-2\epsilon$ dimensions on can show:

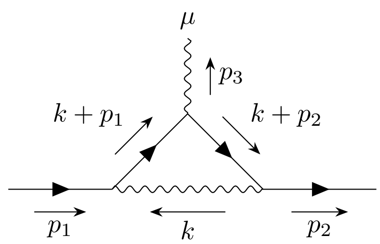


$$= \Big(-g^{\mu\nu}p^2 + p^\mu p^\nu\Big)\left(-i\frac{\alpha}{4\pi}\right)\left(-\frac{4}{3\epsilon} + \mathcal{O}(\epsilon^0)\right). \quad (5.2.25)$$

You can find the complete results, with finite terms, in Mathematica example.

5.2.3 Vertex correction

This computation requires considerable extra work and is necessary to fix $\delta_1^{(1)}$. If we accept the result $Z_1 = Z_2$, then we can avoid the computation, but validating the result is an extremely good cross check. The one-loop diagram is:



$$= -ie_R\mu_R^\epsilon \overline{u}(p_2, m_R)\Gamma^{(1)\mu}u(p_1, m_R) \quad (5.2.26)$$

where we have the conservation of momentum and (in Feynman gauge):

$$\Gamma^{(1)\mu} = -i^3(-ie_R\mu_R^\epsilon)^2\int_k \frac{\gamma^\nu(\not{k}+\not{p}_2+m_R)\gamma^\mu(\not{k}+\not{p}_1+m_R)\gamma_\nu}{D(k,0)D(k+p_1,m_R)D(k+p_2,m_R)}. \quad (5.2.27)$$

A general basis for $\Gamma^{\mu}_{\alpha\beta}(p_1, p_2)$ would have 3 elements $\gamma^{\mu}_{\alpha\beta}, p_1^{\mu}\delta_{\alpha\beta}$ and $p_2^{\mu}\delta_{\alpha\beta}$. However, Γ^{μ} satisfies the **Ward identity**:

$$\Gamma^{\mu}(p_1, p_2)p_{3\mu} = \Gamma^{\mu}(p_1, p_2)(p_1 - p_2)_{\mu} = 0. \tag{5.2.28}$$

As a result, Γ^{μ} can be described in terms of 2 indipendent tensor structures. The vertex descrives the elettromagnetic interaction and so it is useful to use a tensor basis that exposes electric and magnetic properties (we will explore the direct connection later). We can write:

$$\Gamma^{\mu}(p_1, p_2) = F_1\gamma^{\mu} + F_2\frac{i\sigma^{\mu\nu}}{2m_R}(-p_1 + p_2)_{\nu} \tag{5.2.29}$$

where:

$$\sigma^{\mu\nu} = \frac{-i}{2}[\gamma^{\mu}, \gamma^{\nu}] \tag{5.2.30}$$

and F_1, F_2 are the *electric* and *magnetic structure functions* respectively.

Proof of Γ^{μ} decomposition We start with:

$$\overline{u}_2\Gamma^{\mu}u_1 = a_0\overline{u}_2\gamma^{\mu}u_1 + a_1p^{\mu}\overline{u}_2u_1 + a_2p_2^{\mu}\overline{u}_2u_1 \tag{5.2.31}$$

we can change basis to $(p_1 + p_2)_{\mu}$ and $(p_1 - p_2)_{\mu}$, so:

$$\overline{u}_2\Gamma^{\mu}u_1 = a_0\overline{u}_2\gamma^{\mu}u_1 + \tilde{a}_1(p_1 - p_2)^{\mu}\overline{u}_2u_1 + \tilde{a}_2(p_1 + p_2)^{\mu}\overline{u}_2u_1 \tag{5.2.32}$$

contracting with p_3^{μ}:

$$\overline{u}_2\Gamma \cdot p_3u_1 = -\overline{u}_2\Gamma \cdot (p_1 - p_2)u_1 \tag{5.2.33}$$
$$= -a_0\overline{u}_2(\not{p}_1 - \not{p}_2)u_1 - -\tilde{a}_1(p_1 - p_2)^2\overline{u}_2u_1 - \tilde{a}_2(p_1^2 - p_2^2)\overline{u}_2u_1 \tag{5.2.34}$$
$$= 0 \tag{5.2.35}$$

where we used the fact that the particle are on-shell, so the first term is zero because are valid:

$$\overline{u}_2\not{p}_2 = m\overline{u}_2 \tag{5.2.36}$$
$$\not{p}_1u_1 = mu_1 \tag{5.2.37}$$

and the third term is zero since:

$$p_1^2 = p_2^2 = m_R^2. \tag{5.2.38}$$

So we have found:

$$\tilde{a}_1 = 0. \tag{5.2.39}$$

Now we can notice that:

$$p^\mu \overline{u}_2 u_1 = p_\nu g^{\nu\mu} u_2 u_1 \tag{5.2.40}$$

$$= \frac{1}{2} p_\nu u_2 \{\gamma^\mu, \gamma^\nu\} u_1 \tag{5.2.41}$$

$$= \frac{1}{2} u_2 \{\gamma^\mu, \not{p}\} u_1 \tag{5.2.42}$$

hence:

$$(p_1 + p_2)^\mu \overline{u}_2 u_1 = \frac{1}{2} \overline{u}_2 \Big\{\gamma^\mu, \not{p}_1 + \not{p}_2\Big\} u_1 \tag{5.2.43}$$

$$= \frac{1}{2} \overline{u}_2 \Big(\gamma^\mu \not{p}_2 + \not{p}_1 \gamma^\mu + 2 m_R \gamma^\mu\Big) u_1 \tag{5.2.44}$$

and now we can expand the magnetic tensor:

$$\overline{u}_2 \left(\frac{i\sigma^{\mu\nu}}{2m_R}(-p_1 + p_2)_\nu\right) u_1 = \overline{u}_2 \frac{1}{4m_R} \Big[\gamma^\mu, -\not{p}_1 + \not{p}_2\Big] u_1 \tag{5.2.45}$$

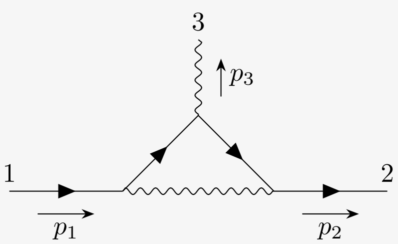
$$= \frac{1}{4m_R} \overline{u}_2 \Big(\gamma^\mu \not{p}_2 + \not{p}_1 \gamma^\mu - 2 m_R \gamma^\mu\Big) u_1 \tag{5.2.46}$$

$$= \frac{1}{2m_R} (p_1 + p_2)^\mu \overline{u}_2 u_1 - \overline{u}_2 \gamma^\mu u_1 \tag{5.2.47}$$

which completes the rearrangement.

The full computation of the vertex diagram is quite long. It's useful to try it in different ways.

Exercise 5. The UV divergence for a triangle topology can be extracted by taking $k \to \infty$ at the integrand level. Compute the resulting integral using Feynman parameters to show that:



$$= -ie_R \mu_R^\epsilon \overline{u}_2 \gamma^\mu u_1 \left(\frac{\alpha}{4\pi}\right) \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0). \tag{5.2.48}$$

Exercise 6. Perform the reduction to scalar integrals using the Passarino-Veltman method to show:

$$F_1^{(1)} = \frac{\alpha}{4\pi} C_\Gamma \left(\frac{1}{\epsilon} + \log\left(\frac{\mu_R^2}{m^2}\right) + \right.$$
$$\left. + \frac{3Q^2 - 8m^2}{Q^2 \beta} \log\left(-\frac{\beta_+}{\beta_-}\right) + \hat{I}_3 \right) + \mathcal{O}(\epsilon) \quad (5.2.49)$$

where:

$$\beta = \sqrt{1 - \frac{4m^2}{Q^2}} \quad , \quad \beta_\pm = \frac{1 \pm \beta}{2} \quad , \quad (-p_1 + p_2)^2 = Q^2 \quad (5.2.50)$$

and:

$$I_{3,0mm}^{(1)[4-2\epsilon]} = \frac{iC_\Gamma}{(4\pi)^2} \frac{1}{4p_1 \cdot p_2} \hat{I}_3. \quad (5.2.51)$$

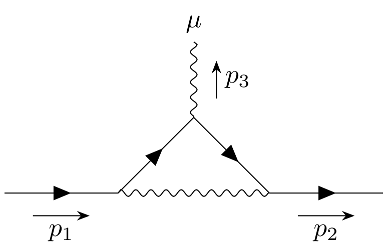
You can also show:

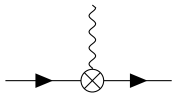
$$F_2^{(1)} = \frac{\alpha}{4\pi} C_\Gamma \frac{4m^2}{Q^2 \beta} \log\left(-\frac{\beta_+}{\beta_-}\right) + \mathcal{O}(\epsilon). \quad (5.2.52)$$

Osservations. We can see in the exercises that $F_2^{(1)}$ is *finite*, and this is consistent with the vertex counter-term which is proportional to γ^μ. The finite terms in $F_1^{(1)}$ are more complicated - in particular the scalar triangle integral which contains an IR divergence. This needs to be treated properly before; an on-shell renormalization will correctly give $Z_1 = Z_2$.

5.2.4 One-loop counterterms

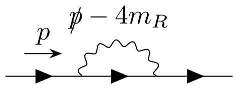
In the $\overline{MS}$ scheme we can now determine the values for $\delta_1, \delta_\psi, \delta_3$ and δ_A. We can see:

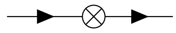


$+$

$= \text{finite} \quad (5.2.53)$

$$\implies \quad \delta_1^{(1)} = \frac{\alpha}{4\pi} C_\Gamma \left(-\frac{1}{\epsilon}\right) \quad (5.2.54)$$

but also:

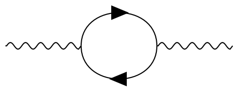


$+$  $=$ finite (5.2.55)

$$\implies \delta_\psi^{(1)} = \frac{\alpha}{4\pi} C_\Gamma \left(-\frac{1}{\epsilon} \right) \tag{5.2.56}$$

$$\implies \delta_3^{(1)} = \frac{\alpha}{4\pi} C_\Gamma \left(-\frac{4}{\epsilon} \right) \tag{5.2.57}$$

and:


$+$  $=$ finite (5.2.58)

$$\implies \delta_A^{(1)} = \frac{\alpha}{4\pi} C_\Gamma \left(-\frac{4}{3\epsilon} \right). \tag{5.2.59}$$

From the constants we may obtain (since $\delta_1 = \delta_\psi$):

$$\delta_e^{(1)} = -\frac{1}{2}\delta_A^{(1)} = \left(\frac{\alpha}{4\pi} \right) C_\Gamma \left(\frac{2}{3\epsilon} \right) \tag{5.2.60}$$

and:

$$\delta_m^{(1)} = \delta_3^{(1)} - \delta_\psi^{(1)} = \left(\frac{\alpha}{4\pi} \right) C_\Gamma \left(-\frac{3}{\epsilon} \right). \tag{5.2.61}$$

On-shell renormalization The Dyson renormalization electron propagator is:

$$\frac{1}{\not{p} - m_R + \Sigma_R(\not{p}, m_R)} \tag{5.2.62}$$

and we would like to ensure the propagator has *no quantum corrections*, so $\Sigma_R = 0$, at the physical pole, so $m_R = m_{phys}$.

We may expand the self-energy Σ_R around $\not{p} = m_R \mathbb{1}$ to obtain:

$$\Sigma_R(\not{p}, m_R) = \Sigma_R(\not{p}, m_R)\Big|_{\not{p}=m_R} + (\not{p} - m_R)\Sigma_R^{WF}(p^2, m_R^2) + \mathcal{O}\big((\not{p} - m_R)^2\big) \tag{5.2.63}$$

where:

$$\Sigma_R^{WF}(p^2, m_R^2) = \frac{\mathrm{d}}{\mathrm{d}\not{p}} \Sigma_R(\not{p}, m_R)\Big|_{\not{p}=m_R}. \tag{5.2.64}$$

So to completely fix δ_ψ and δ_m in the on-shell scheme we must impose:

$$\Sigma_R(\not{p}, m_R)\Big|_{\not{p}=m_R} = 0 \tag{5.2.65}$$

$$\frac{\mathrm{d}}{\mathrm{d}\not{p}} \Sigma_R(\not{p}, m_R)\Big|_{\not{p}=m_R} = 0. \tag{5.2.66}$$

the photon counterterm can be fixed at zero momentum:

$$\lim_{p^2\to 0}\Big(\Pi(p^2, m_R^2)\Big) = 0 \tag{5.2.67}$$

and the vertex correction can be fixed to a mesurement at low Q^2 (i.e. Thomson scattering):

$$\lim_{Q^2\to 0}\Big(\Gamma_R^\mu(Q^2, m_R^2)\Big) = -ie_{phys}\gamma^\mu \tag{5.2.68}$$

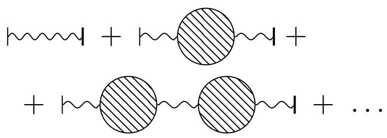
and note that low Q^2 implies that we are far away from interaction point.

There is a subtelty regarding the on-shell scheme renormalization constants in that the finite part of the vertex correction had an IR divergence - the wave function renormalization condition (5.2.66) also contains an IR divergence, such that we obtain $Z_\psi = Z_1$ as expected.

We complete this section with a couple of loose end.

Resummation of photon propagator For $\xi_R = 1$ the (Fourier tranformed) 2-photon Green function is:

$\tilde{G}_{\mu\nu}(p^2) =$



(5.2.69)

$$\begin{aligned} &= \frac{-ig_{\mu\nu}}{p^2} + \frac{-ig_{\mu\sigma}}{p^2}\Pi^{\sigma\rho}\frac{-ig_{\rho\nu}}{p^2} + \\ &+ \frac{-ig_{\mu\sigma}}{p^2}\Pi^{\sigma\rho}\frac{-ig_{\rho\gamma}}{p^2}\Pi^{\gamma\delta}\frac{-ig_{\delta\nu}}{p^2} + \dots \end{aligned} \tag{5.2.70}$$

If we take:

$$\Pi^{\mu\nu} = i\Big(p^2 g^{\mu\nu} - p^\mu p^\nu\Big)\Pi \tag{5.2.71}$$

we get:

$$\begin{aligned} \tilde{G}_{\mu\nu} &= \frac{-ig_{\mu\nu}}{p^2} + \frac{i}{p^4}\Big(p^2 g_{\mu\nu} - p_\mu p_\nu\Big)\Pi - \\ &- \frac{i}{p^6}\underbrace{\Big(p^2 g_{\mu\gamma} - p_\mu p_\gamma\Big)\Big(p^2 g_\nu^\gamma - p^\gamma p_\nu\Big)}_{=p^2\left(p^2 g_{\mu\nu} - p_\mu p_\nu\right)}\Pi^2 + \dots \end{aligned} \tag{5.2.72}$$

$$= \frac{-i}{p^4}\Big(p^2 g_{\mu\nu} - p_\mu p_\nu\Big)\Big(1 - \Pi + \Pi^2 + \dots\Big) - i\frac{p_\mu p_\nu}{p^4} \tag{5.2.73}$$

so we can resum the Π and get (summing and subtracting the same term):

$$\tilde{G}_{\mu\nu} = \frac{-i}{p^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{1}{1+\Pi}\underbrace{-i\frac{p_\mu p_\nu}{p^4}}_{\tilde{G}_{\mu\nu,L}} \tag{5.2.74}$$

so we see that only the transverse part of the photon propagator is affected. The longitudinal component, $\tilde{G}_{\mu\nu,L}$, does not contribute to physical quantities thanks to the Ward identity.

Proof that $Z_\xi = 1$ We can show that gauge invariance dictates $Z_\xi = 1$ by studying the 2-photon Green function:

$$\tilde{G}_{0,\mu\nu}(p^2, \xi_0) = Z_A \tilde{G}_{R,\mu\nu}(p^2, \xi_R) \tag{5.2.75}$$

where we can write:

$$\tilde{G}_{0,\mu\nu}(p^2, \xi_0) = -\frac{i}{p^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{1}{1+\Pi_0} - i\xi_0 \frac{p_\mu p_\nu}{p^4} \tag{5.2.76}$$

$$Z_A \tilde{G}_{R,\mu\nu}(p^2, \xi_R) = Z_A \left(\frac{-i}{p^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{1}{1+\Pi_R} - i\xi_R \frac{p_\mu p_\nu}{p^4}\right) \tag{5.2.77}$$

where we can see the coefficient of $g^{\mu\nu}$:

$$\frac{1}{1+\Pi_0} = \frac{Z_A}{1+\Pi_R} \tag{5.2.78}$$

and the coefficient of $p^\mu p^\nu / p^2$ minus the coefficient of $g^{\mu\nu}$:

$$\xi_0 = Z_A \xi_R \tag{5.2.79}$$

but by the definition:

$$\xi_0 = \frac{Z_A}{Z_\xi}\xi_R \tag{5.2.80}$$

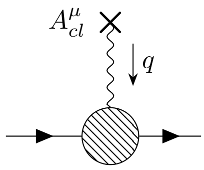
so we are only consistent if:

$$Z_\xi = 1. \tag{5.2.81}$$

5.3 The interpretation of quantum corrections to F_1 and F_2

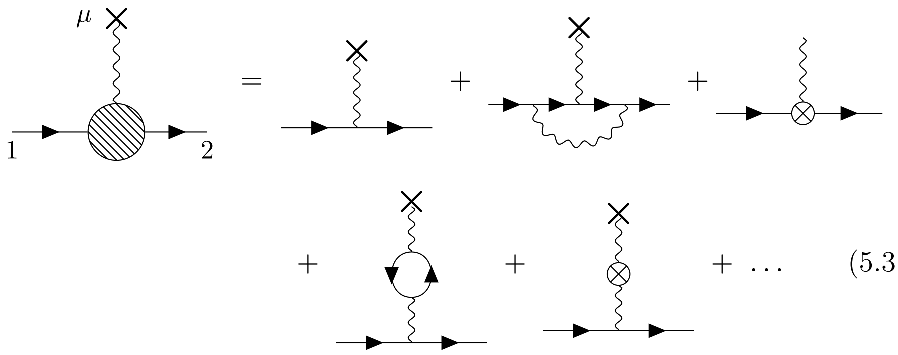
The reference for this part is the Itzykson-Zuber [4] at page 347-349.

In this section we show the direct connection between electric and magnetic interactions and the structure functions F_1, F_2, by considering the interaction between and electron and a slowly varying semi-classical photon field A^μ_{cl}:



$$= -ie_R \mu_R^\epsilon \overline{\psi}\gamma_\mu \psi A^\mu_{cl} + \dots \tag{5.3.1}$$

By taking the limit $q^2 \to 0$, and the slow variation ($q^\mu \sim \frac{\partial}{\partial x^\mu}$) we can expand the interaction as follows:



(5.3.2)

$$= -ie_R\mu_R^\epsilon \overline{u}_2 \Big(\gamma_\mu + \Gamma^{(1)}_{R\mu}(q^2, m_R^2) + \tag{5.3.3}$$
$$+ \gamma_\nu D^{\nu\rho}(q^2)\Pi^{(1)}_{R\,\rho\mu}(q^2, m_R^2)\Big) u_1 A^\mu_{cl} + \mathcal{O}(e^5)$$

$$= \tilde{J}_\mu A^\mu_{cl}. \tag{5.3.4}$$

The interaction is described by an effective hamiltonian:

$$\Delta H_{int} = \int \mathrm{d}^3x\, J_\mu A^\mu_{cl} \tag{5.3.5}$$

where J_μ is the Fourier transform of $\tilde{J}_\mu$. Let's look at $\overline{u}_2\Gamma^{(1)}_{R,\mu}u_1$ first:

$$\overline{u}_2\Gamma^{(1)}_{R,\mu}u_1 = \overline{u}_2\gamma_\mu u_1 F^{(1)}_{1,R}(q^2, m_R^2) + \frac{i}{2m}\overline{u}_2\sigma_{\mu\nu}u_1(p_1+p_2)^\nu F^{(1)}_{2,R}(q^2, m_R^2) \tag{5.3.6}$$

where we can show:

$$F^{(1)}_{1,R}(q^2, m_R^2) \overset{q^2\to 0}{\longrightarrow} \left(\frac{\alpha}{4\pi}\right)\frac{q^2}{m_R^2}\Big(a + b(\text{IR divergence})\Big) \tag{5.3.7}$$

$$F^{(1)}_{2,R}(q^2, m_R^2) \overset{q^2\to 0}{\longrightarrow} \left(\frac{\alpha}{4\pi}\right) 2. \tag{5.3.8}$$

Note. Renormalization has explicitly removed leading terms $(q^2/m_R^2)^0$ in the expansion of F_1.

We may also look at the second term propagator to $\Pi^{(1)}$ in the same limit, and find:

$$\gamma_\nu D^{\nu\rho}(q^2)\Pi^{(1)}_{R,\rho\mu}(q^2, m_R^2) \overset{q^2\to 0}{\longrightarrow} \left(\frac{\alpha}{4\pi}\right)\gamma_\mu\frac{q^2}{m_R^2}C \tag{5.3.9}$$

where C is a constant. In position space we have:

$$\frac{q^2}{m_R^2} \longrightarrow \frac{\Box}{m_R^2} \tag{5.3.10}$$

so having examined the elements of $\tilde{J}_\mu$ we may return to the expression for ΔH_{int} and show:

$$\Delta H_{int} \propto e_R \int \mathrm{d}^3x \left\{ \overline{\psi} \overleftrightarrow{\partial}_\mu \psi \left(1 + \left(\frac{\alpha}{4\pi}\right) \left(a + C + b(\text{IR divergence})\right) \Box \right) A^\mu_{cl} + \right.$$
$$\left. + \left(\frac{1}{2} + \left(\frac{\alpha}{4\pi}\right)\right) \frac{1}{2m_R} \overline{\psi} \sigma^{\mu\nu} \psi F_{cl,\mu\nu} \right\}$$

where we have:

$$F_{cl,\mu\nu} = \partial_\mu A_{cl,\nu} + \partial_\nu A_{cl,\mu}. \tag{5.3.11}$$

Hint. One must use that terms with more than 2 derivatives vanish due to the slow variation condtion of $A_{cl,\mu}$ and apply to Gordon Identity.

Remarks. The first part is clearly identified as the electric current. The second term is the magnetic moment of the electron.

Recall $F^{ij} = \epsilon^{ijk} B_k$ and so:

$$\Delta H_{int} \sim (\text{electric current}) - \vec{B} \cdot \left[\frac{e_R}{2m_R} \left(1 + \frac{\alpha}{2\pi}\right) 2 \int \mathrm{d}^3x \, \overline{\psi} \frac{\sigma}{2} \psi \right] \tag{5.3.12}$$

where:

$$\vec{\mu} = \left[\frac{e_R}{2m_R} \left(1 + \frac{\alpha}{2\pi}\right) 2 \int \mathrm{d}^3x \, \overline{\psi} \frac{\sigma}{2} \psi \right] \tag{5.3.13}$$

is the *magnetic moment.* From the Dirac equation (semi-classical) one finds:

$$\vec{\mu}_{Dirac} = \frac{e}{2m} (2\vec{s}) \tag{5.3.14}$$

$$= g_{Dirac} \left(\frac{e}{2m} \frac{\vec{\sigma}}{2} \right) \tag{5.3.15}$$

where $\vec{s}$ is the spin operator and $g_{Dirac} = 2$. The first order correction to g is therefore:

$$g - 2 = \frac{\alpha}{2\pi} \tag{5.3.16}$$

which is Schwinger's famous result (1948).

5.4 Interpretation of $\Pi(p^2, m^2)$ - the vacuum polarisation

The renormalization 2-point function $\Pi_R(p^2, m_R^2)$ has inplications for the range of the elettromagnetic interaction.

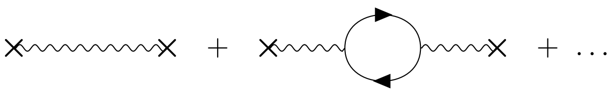
If we consider a static photon:

$$q^0 = 0 \implies q^2 = -|\vec{q}|^2 \tag{5.4.1}$$

then for $|\vec{q}|^2 \ll 1$ we have:

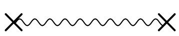
$$\Pi_R(-|\vec{q}|^2, m_R^2) \xrightarrow{|\vec{q}|^2 \to 0} \frac{4}{15} \frac{|\vec{q}|^2}{m_R^2} + \mathcal{O}\left(\frac{|\vec{q}|^4}{m_R^4}\right). \tag{5.4.2}$$

The Dyson resummed propagator gave us:



(5.4.3)

where we have used a cross at the end points to show we are propagating between two points far apart, i.e. $|\vec{q}|^2 \to 0$:

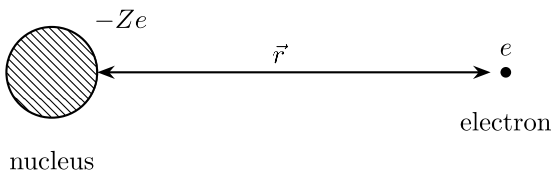


$$= \lim_{|q|^2 \to 0} \frac{(-ie_R)^2}{-|\vec{q}|^2} = \frac{e_R^2}{|\vec{q}|^2} \tag{5.4.4}$$

so the Dyson resummed expression is:

$$\frac{e_R^2}{|\vec{q}|^2}\left(\frac{1}{1+\Pi_R(-|\vec{q}|^2, m_R^2)}\right) = \frac{e_R^2}{|\vec{q}|^2}\left(1 + \left(\frac{\alpha}{4\pi}\right)\frac{4}{15}\frac{|\vec{q}|^2}{m_R^2} + \mathcal{O}(\alpha^2)\right). \tag{5.4.5}$$

We can use this result to read off the potential betwwen a heavy nucleus (charge $-Ze$) and an electron:



$$V(r) = \frac{-Ze_R^2}{4\pi|\vec{r}|}\left(1 + \left(\frac{\alpha}{4\pi}\right)\frac{4}{15}\frac{\Delta(\vec{r})}{m_R^2} + \mathcal{O}(\alpha^2)\right). \tag{5.4.6}$$

The correction to $V(r)$ is called the Uehling potential and results in a charge screening effect. We have that $\Delta(\vec{r}) \approx \delta^{(3)}(\vec{r})$ but can be computed exactly to be a distribution concentrated in a region:

$$r \leq \frac{1}{m} \sim \lambda - \text{the Compton wavelength.}$$

So, for $r \geq 1/m$ the vistual e^+e^- pairs accounted for in Π_R effect the medium around the electron, reducing the effective electric charge. This effect is known as *vacuum polarisation*, see figure 5.1, and the reference is Peskin and Schroeder ad p.255 figure 7.8.

We can also consider the effective coupling at very small distance by taking $|\vec{q}|^2 \gg m_R^2$ in which gives:

$$\Pi_R(-|\vec{q}|^2, m_R^2) \xrightarrow{|\vec{q}|^2 \gg m_R^2} \left(\frac{\alpha}{4\pi}\right)\frac{4}{q}\left(-5 + 3\log\left(\frac{|q|^2}{m_R^2}\right)\right) + \mathcal{O}\left(\frac{m^2}{|q|^2}\right) \tag{5.4.7}$$

$$\implies \quad \alpha_{eff}(|\vec{q}|^2) \overset{|\vec{q}|^2 \gg m_R^2}{=} \frac{\alpha}{1 - \frac{\alpha}{3\pi}\log\left(\frac{|\vec{q}|^2}{Am_R^2}\right)} \tag{5.4.8}$$

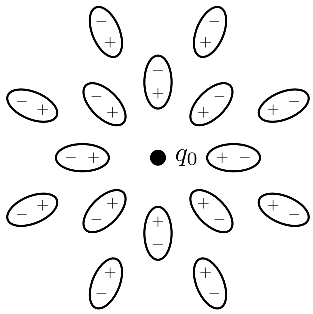


Figure 5.1

where:

$$A = \exp\left\{ \left(\frac{5}{3} \right) \right\}. \tag{5.4.9}$$

So, we get $\alpha_{eff} < \alpha$ at small distance.

5.5 The Ward-Takahashi identity

We have already encountered the *Ward Identity* of the form:

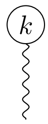
$$\mathcal{A}^{\mu} p_{\mu} = 0 \tag{5.5.1}$$

for an on-shell amplitude $\mathcal{A} = \mathcal{A}^{\mu}\epsilon_{\mu}(p)$. There is a stronger statement that applies to off-shell correlation functions from which the Ward identity follows. This is the **Ward-Takahashi identity**.

Take a correlation function with n e^+e^- pairs and m photons:

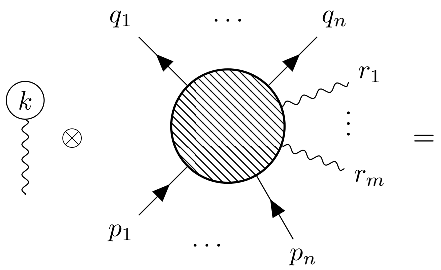
$$\mathcal{M}^{\mu}(p_1, \ldots, p_n; q_1, \ldots, q_n; r_1, \ldots, r_m)$$

where p_i are incoming fermion, q_i are outgoing fermions and r_i are outgoing photons. An incoming photon with momentum k^{μ} is inserted at all point along the e^+e^- currents, denoted as:

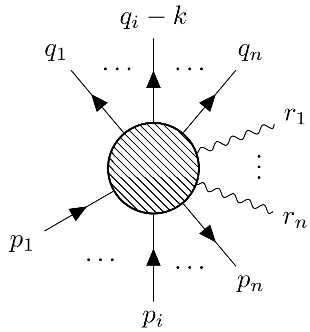


$$\otimes\,. \tag{5.5.2}$$

The Ward-Takahashi identity states:



$$= e_r \sum_{i=1}^{n}$$



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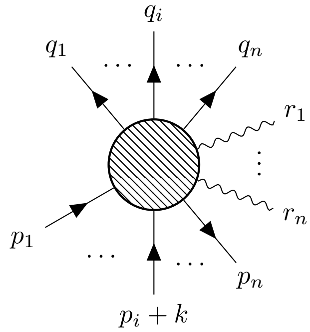
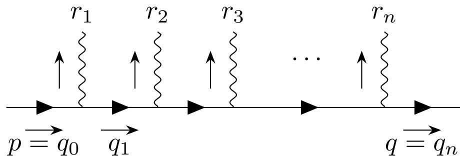


. (5.5.3)

We can prove this relation to all orders using Feynman diagrams, demostrating that it relies on cancellations between the diagrams.

Proof To start the proof we consider a fermion line with n photons attached:

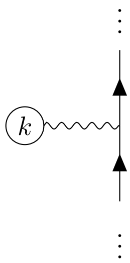


where we have:

$$q_1 = q_0 - r_1 \tag{5.5.4}$$

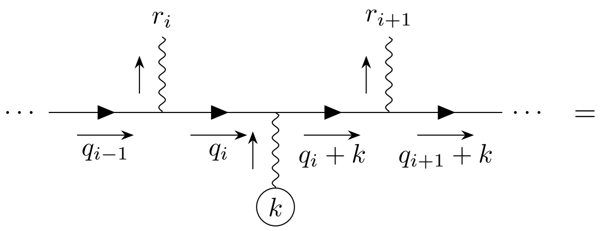
$$q = q_n = q_0 - \sum_{i=1}^{n} r_i. \tag{5.5.5}$$

We then add on insertion of a new photon momenta k^μ denoted as:



$$= \not{k}. \tag{5.5.6}$$

If we make the insertion, between photons i and $i+1$, we will see:



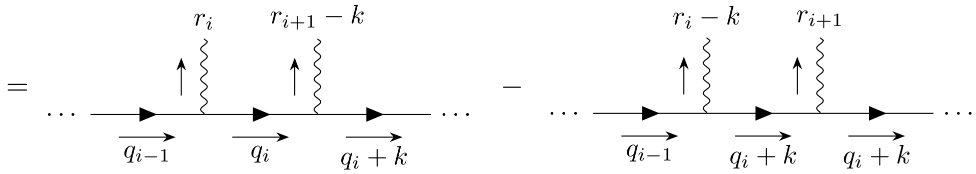
$$= \gamma^{\mu_i} \frac{\not q_i + m}{q_i^2 - m^2} \not k \frac{\not q_i + \not k + m}{(q_i + k)^2 - m^2} \gamma^{\mu_{i+1}} e_R \quad (5.5.7)$$

where we can write:

$$\not k = (\not q + \not k - m) - (\not q_i - m) \quad (5.5.8)$$

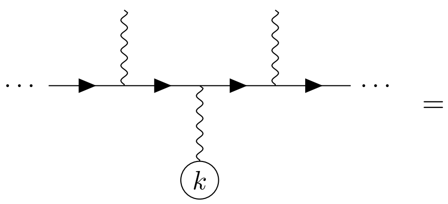
so we get:

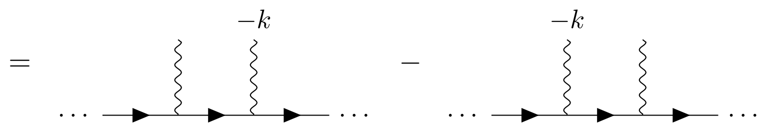
$$= \gamma^{\mu_i} \left(\frac{\not q_i + m}{q_i^2 - m^2} - \frac{\not q_i + \not k + m}{(q_i + k)^2 - m^2} \right) \gamma^{\mu_{i+1}} \quad (5.5.9)$$



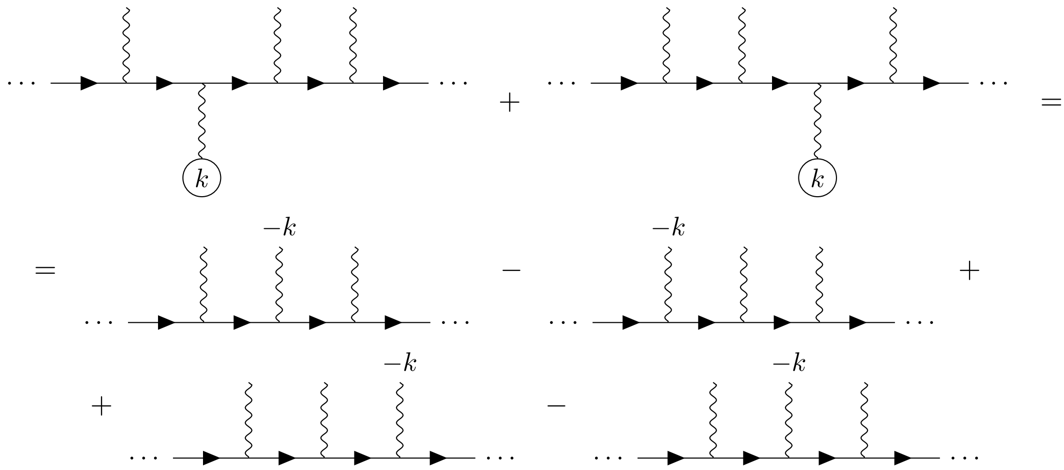
(5.5.10)

using a shorthand notation:

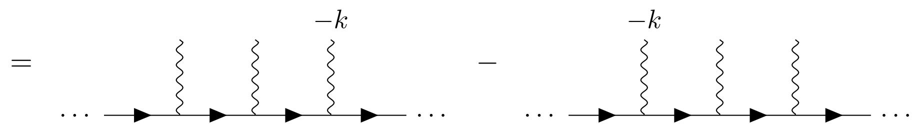




we can sum insertions on adjacent propagators. We can write:

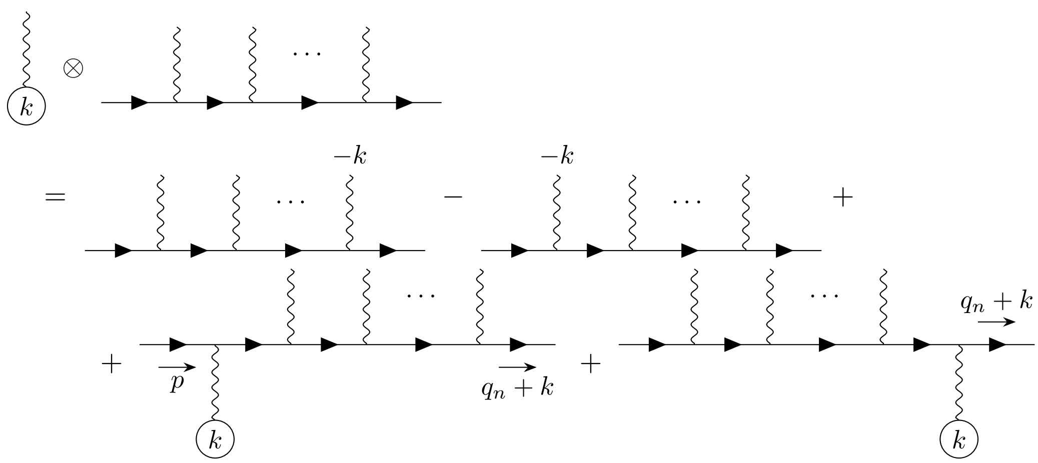


(5.5.11)



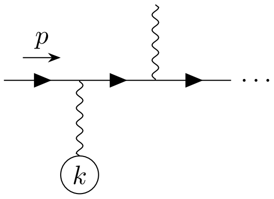
(5.5.12)

so we get (where with $\otimes$ we mean the sum over all insertions):

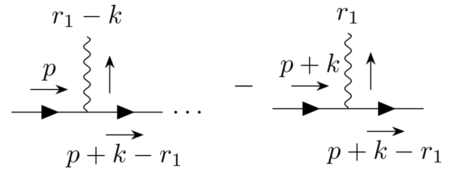


(5.5.13)

also we have:

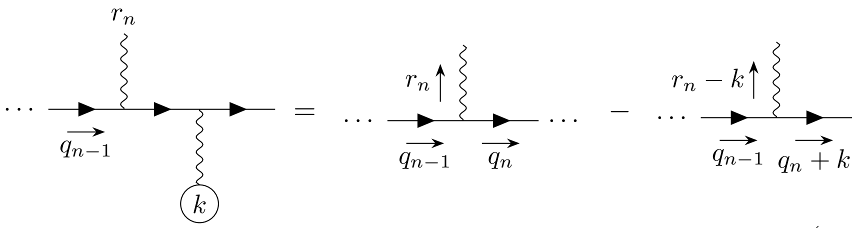


$$= \frac{(\not{p}+m)}{p^2-m^2}\not{k}\frac{(\not{p}+\not{k}+m)}{(p+k)^2-m^2} \tag{5.5.14}$$



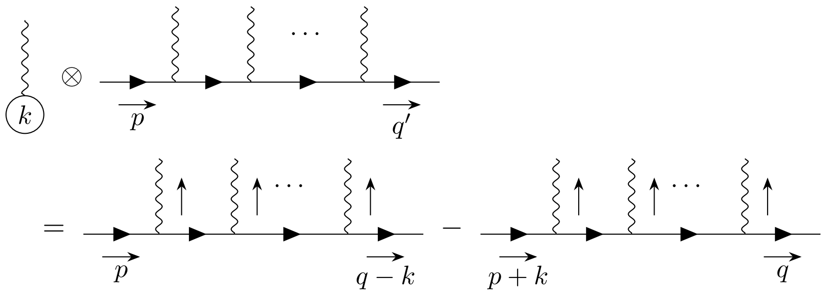
(5.5.15)

and:



(5.5.16)

So we get:



(5.5.17)

where:

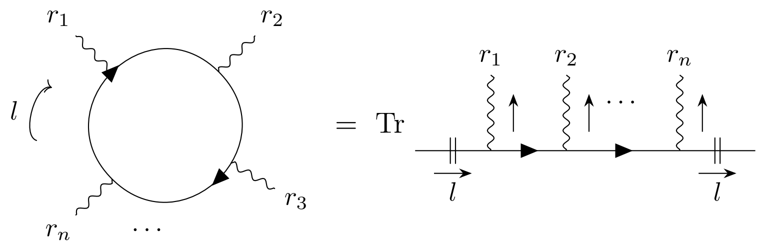
$$q = p + k - \sum_i r_i \tag{5.5.18}$$

$$q' = p - \sum_i r_i. \tag{5.5.19}$$

Note that if we apply the LSZ formalism to the correlation function and put the external legs on-shell, the two terms cancel, recovering the Ward identity.

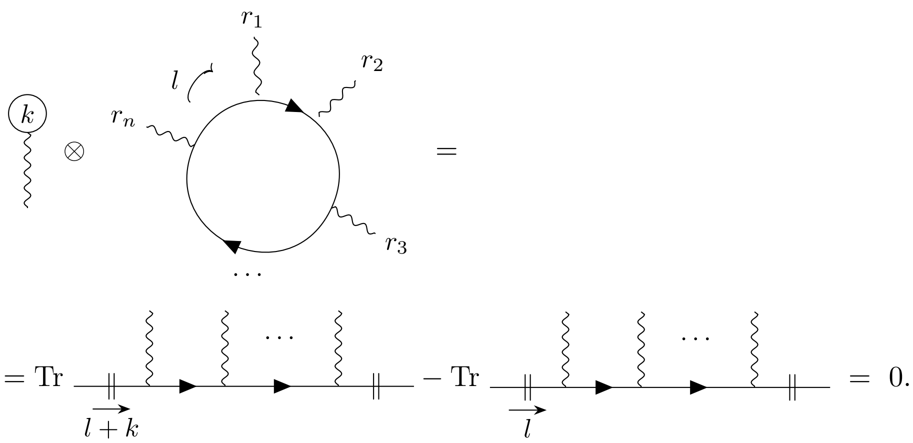
The relation is valid off-shell, so applies to any sub-diagram over a more complicated multi-loop diagram. To complete the proof we must consider insertions on internal closed loop, fermion lines.

A closed loop with n-photons attached is written as:



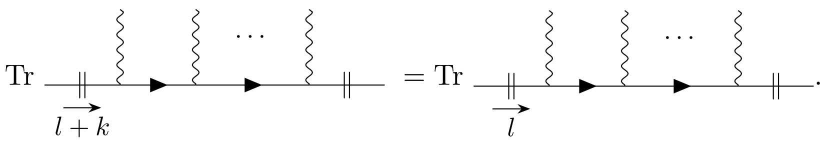
(5.5.20)

where r_i are outgoing photon momenta and l is an internal loop momentum. On the left hand side we have used to notation —‖— to indicate the end-points are the same. When we sum over all insertion point we can use the same type of relations as before, only the changed momentum conservation conditions results in the final two contributions cancelling. We have:



(5.5.21)

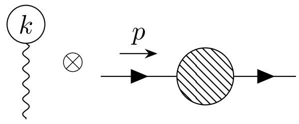
Equation (5.5.21) is verified performing the shift $l \to l - k$, with we have:



(5.5.22)

Equations (5.5.17) and (5.5.21) are sufficient to prove the general relation.

It is interesting to consider a special case of the Ward-Takahashi identity for the 2-point electron self energy:



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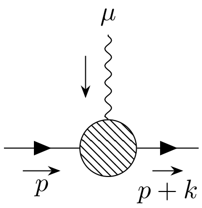
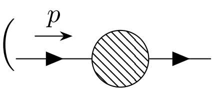
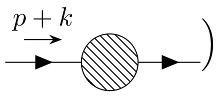


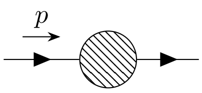
(5.5.23)

$= e_R \Big($ 

$-$ 

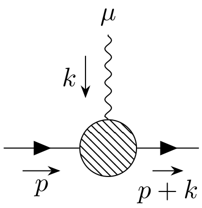
$\Big).$ (5.5.24)

The two point function is:



$$= \frac{i}{\not{p} - m_R - \Sigma_R(p, m_R)} = S(p) \tag{5.5.25}$$

while the vertex function appears on the lefthand side as:



$$= S(p)\Big(- ie_R \Gamma^\mu_R(p, p+k) k_\mu \Big) S(p+k) \tag{5.5.26}$$

remembering to add external propagators for the correlation function.

Multiplying the inverse propagators leads to the relation:

$$-ik_\mu \Gamma^\mu_R(p, p+k) = S^{-1}(p+k) - S^{-1}(p) \tag{5.5.27}$$

as we take the $k \to 0$ limit we identify the renormalization constants Z_1 and Z_ψ:

$$\Gamma^\mu_R(p, p+k) \overset{k\to 0}{\longrightarrow} Z_1^{-1}\gamma^\mu \tag{5.5.28}$$

$$S(p) = \frac{iZ_\psi}{\not{p} - m_R} + \mathcal{O}(\not{p} - m_R) \tag{5.5.29}$$

hence:

$$-i\not{k} Z_1^{-1} = -i(\not{p} + \not{k}) Z_\psi^{-1} - (-i\not{p}) Z_\psi^{-1} \tag{5.5.30}$$

that implies:

$$Z_1 = Z_\psi. \tag{5.5.31}$$

So the Ward-Takahashi identity is directly connected with the gauge invariance constraint, that identifies Z_1 and Z_2.

Chapter 6

Renormalization of QCD at one-loop

In this chapter we will show how too organize $SU(N)_C$ colour algebra, we will introduce the spinor-helicity formalism to find compact representations for on-shell helicity amplitudes. There will be an hidden simplicity in QCD at tree-level.

After that we will perform the computations to fix $\delta^{(1)}_{\psi_g}$, $\delta^{(1)}_{m_g}$, $\delta^{(1)}_{A}$ and $\delta^{(1)}_{1}$ sufficient to fix $\delta^{(1)}_{g_S}$. We will see a sign flip respect the result of QED.

At the end we will analyze further divergent graphs that can be used to validate structure.

6.1 Lagrangian and Feynman rules

We will consider an $SU(N_c)$ gauge theory with n_f *quark flavour* trasforming in the fundamental representation. The *gluon field* A^a_μ is quantized in a covariant gauge using the Fadeev-Popov method which introduces the ghost fields c^a, that are scalar satisfing fermionic statistics. The complete field content is:

$$
\begin{aligned}
&\psi_{q\alpha i} \quad \text{quark field} \quad , && q = 1, \ldots, n_f \text{ (flavour)} \\
& && \alpha = 1, \ldots, 4 \text{ (spinor)} \\
& && \alpha = 1, \ldots, N_c \text{ (color)} \\
&A^{\mu a} \quad \text{gluon field} \quad , && \mu = 0, \ldots, 3 \text{ (Lorentz)} \\
& && a = 1, \ldots, N_c^2 - 1 \text{ (color)} \\
&c^a \quad \text{ghost field} \quad , && a = 1, \ldots, N_c^2 - 1 \text{ (color).}
\end{aligned}
$$

The bare lagrangian is:

$$\mathcal{L} = -\frac{1}{4} F_0^{\mu\nu a} F^a_{0\mu\nu} + \sum_{q=1}^{n_f} \overline{\psi}_{0q\alpha i} \left(i\not{\mathcal{D}}_0 - m_{0q} \right)_{ij\alpha\beta} \psi_{0qj\beta} - \\ - \frac{1}{2\xi} \left(\partial_\mu A_0^{\mu a} \right) \left(\partial_\nu A_0^{\nu a} \right) + \partial^\mu \overline{c}_0^a \mathcal{D}^{ab}_{0\mu} c_0^b \tag{6.1.1}$$

where:

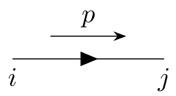
$$F_0^{\mu\nu a} = \partial^\mu A_0^{\nu a} - \partial^\nu A_0^{\mu a} - g_{s0} f^{abc} A_0^{\mu b} A_0^{\nu c} \tag{6.1.2}$$

$$\not{\mathcal{D}}^\mu_{0ij\alpha\beta} = \gamma^\mu_{\alpha\beta} \mathcal{D}_{0\mu ij} \tag{6.1.3}$$

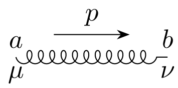
$$\mathcal{D}^\mu_{0ij} = \partial^\mu \delta_{ij} - i g_{s0} t^a_{ij} A_0^{\mu a} \tag{6.1.4}$$

$$\mathcal{D}^{\mu ab} = \partial^\mu \delta^{ab} - g_{s0} f^{abc} A^{\mu c}. \tag{6.1.5}$$

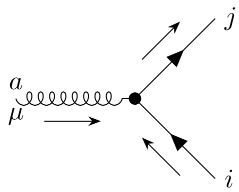
with f^{abc} the $SU(N_c)$ *structure constants*, g_{s0} the *bare strong coupling constant* and t^a_{ij} the fundamental $SU(N_c)$ *generators*. We can also remember the Feynman rules, using implicit spinor indices and subscripts for bare quantities:



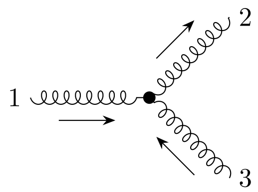
$$\frac{i(\not{p} + m_q)\delta_{ij}}{D(p, m_q)} \tag{6.1.6}$$



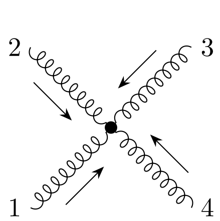
$$\frac{-i\delta^{ab}}{D(p, m_q)} \left[g_{\mu\nu} - (1-\xi) \frac{p_\mu p_\nu}{p^2} \right] \tag{6.1.7}$$



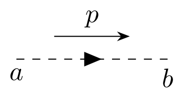
$$i g_s t^a_{ij} \gamma^\mu \tag{6.1.8}$$



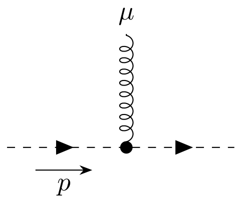
$$-g_s f^{a_1 a_2 a_3} \Big[g^{\mu_1 \mu_2} (p_1 - p_2)^{\mu^3} + g^{\mu_2 \mu_3} (p_2 - p_3)^{\mu_1} + \\ + g^{\mu_3 \mu_1} (p_3 - p_1)^{\mu_2} \Big] \tag{6.1.9}$$



$$i g_s^2 \Big[f^{12b} f^{b34} (g^{13} g^{24} - g^{14} g^{23}) + \\ + f^{13b} f^{b24} (g^{12} g^{34} - g^{14} g^{23}) + \\ + f^{14b} f^{b23} (g^{13} g^{24} - g^{12} g^{34}) \Big] \tag{6.1.10}$$



$$\frac{i\delta^{ab}}{D(p,0)} \tag{6.1.11}$$



$$g_s f^{abc} p_\mu . \tag{6.1.12}$$

6.1.1 Color algebra

$SU(N_c)$ color factors are a new feature of QCD diagrams. We must simplify them using $SU(N_c)$ color algebra:

$$\mathrm{Tr}\Big\{t^a t^b\Big\} = \frac{1}{2}\delta^{ab} \tag{6.1.13}$$

$$\mathrm{Tr}\{t^a\} = 0 \tag{6.1.14}$$

$$\Big[t^a, t^b\Big] = i f^{abc} t^c \tag{6.1.15}$$

all of them bring to:

$$f^{abc} = -2i\,\mathrm{Tr}\Big\{\Big[t^a, t^b\Big]t^c\Big\}. \tag{6.1.16}$$

Note that the general relation for (6.1.13) is:

$$\mathrm{Tr}\Big\{t^a t^b\Big\} = T_R \delta^{ab}.$$

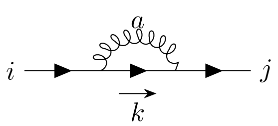
We can use the basic properties to prove the **Fierz identity**:

$$t^a_{ij} t^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right) \tag{6.1.17}$$

or the **Jacobi identity**:

$$f^{12x} f^{x34} + f^{13x} f^{x42} + f^{14x} f^{x23} = 0. \tag{6.1.18}$$

By repeted use of these properties we may reduce onto a basis of color factors. For example, consider to color factor for the quark self-energy that we will need later:



$$= t^a_{jk} t^a_{ki} \tag{6.1.19}$$

$$= \frac{1}{2}\left(\delta_{ij}\delta_{kk} - \frac{1}{N_c}\delta_{jk}\delta_{ki}\right) \tag{6.1.20}$$

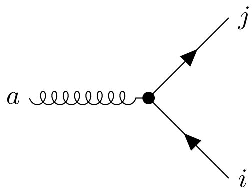
$$= \frac{1}{2}\left(N_c - \frac{1}{N_c}\right)\delta_{ij} \tag{6.1.21}$$

$$= C_F \delta_{ij} \tag{6.1.22}$$

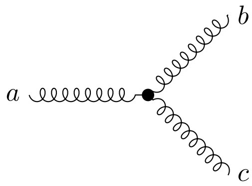
where we identified the fundamental Casimir of $SU(N_c)$:

$$C_F = \frac{N_c^2 - 1}{2N_c}. \tag{6.1.23}$$

It is possible to apply the color algebra graphically as well. If we use the Feynman diagram to represent only the color factor then:



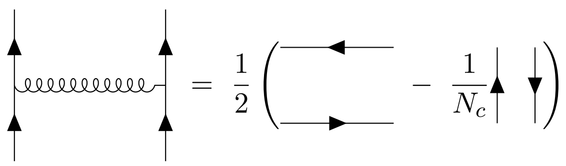
$$= t^a_{ji} \tag{6.1.24}$$



$$= f^{abc} \tag{6.1.25}$$

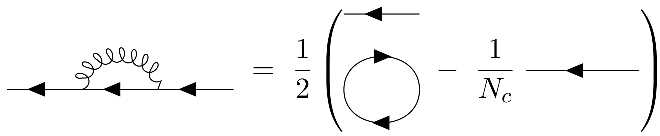
etc. (6.1.26)

The Fierz identity then reads:



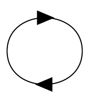
(6.1.27)

from which we can connect the lower quark indice to make the self energy color factor:



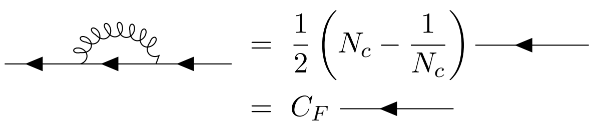
(6.1.28)

where:



$$= N_c \tag{6.1.29}$$

so:

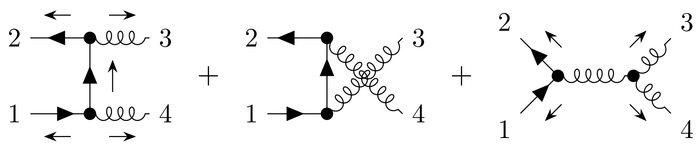


(6.1.30)
(6.1.31)

6.1.2 Tree-level amplitudes and spinor-helicity method

Having arranged color factors it is also worth noting how we can compute compact helicity amplitudes for the remaining kinematic components. We will introduce the concepts by means of an example: $\bar{q}qgg$ scattering. There are 3 tree-level Feynamn diagrams ($m_q = 0$):

$\mathcal{A} =$



(6.1.32)

naming respectively 1, 2, 3, and where we have chosen all particle out-going to simplify momentum conservation:

$$\sum_{i=1}^{4} p_i^\mu = 0. \tag{6.1.33}$$

Firstly, let's show that the amplitude satisfies the Ward identity:

$$\mathcal{A}^\mu p_{3\mu} = 0 \tag{6.1.34}$$

where:

$$\mathcal{A} = \mathcal{A}^\mu \epsilon_\mu(p_3). \tag{6.1.35}$$

The first diagram reads:

$$\mathcal{A}_1 = i(ig_s)^2 \overline{u}_1 \not{\epsilon}_4 \frac{\not{p}_{23}}{s_{23}} \not{\epsilon}_3 v_2 \left(t^{a_3} t_{a_4}\right)_{i_2 i_1} \tag{6.1.36}$$

where $p_{ij} = p_i + p_j$ and $s_{23} = 2p_2 \cdot p_3$. The contribution to the Ward identity, defining $C_1 = (t^{a_3} t_{a_4})_{i_2 i_1}$ is then:

$$\mathcal{A}_1^\mu p_{3\mu} = -ig_s^2 \overline{u}_1 \not{\epsilon}_4 \frac{\not{p}_{23}}{s_{23}} \not{p}_3 v_2 C_1 \tag{6.1.37}$$

$$= -ig_s^2 \overline{u}_1 \not{\epsilon}_4 \frac{\not{p}_2 \not{p}_3}{s_{23}} v_2 C_1 \tag{6.1.38}$$

$$= -ig_s^2 \overline{u}_1 \not{\epsilon}_4 v_2 C_1 \tag{6.1.39}$$

using in the last step the Clifford algebra. The second diagram is related to the first one through $3 \leftrightarrow 4$ and we find:

$$\mathcal{A}_2^\mu p_{3\mu} = ig_s^2 \overline{u}_1 \not{\epsilon}_4 v_2 C_2 \quad , \quad C_2 = \left(t^{a_4} t^{a_3}\right)_{i_2 i_1}. \tag{6.1.40}$$

The third diagram is the most complicated:

$$\mathcal{A}_3 = g_s \overline{u}_1 \gamma^\mu v_2 \frac{1}{s_{12}} \hat{V}_{3g\mu\mu_3\mu_4}(-p_{34}, p_3, p_4) \epsilon_3^{\mu_1} \epsilon_4^{\mu_4} t^b_{i_1 i_2} f^{ba_3a_4} \tag{6.1.41}$$

$$= g_s^2 t^b_{i_1 i_2} f^{ba_3a_4} \frac{1}{s_{12}} \overline{u}_1 \gamma^\mu v_2 \Big(\epsilon_3 \cdot \epsilon_4 (p_3 - p_4)_\mu + \epsilon_{4\mu} 2p_4 \cdot \epsilon_3 - \epsilon_{3\mu} 2p_3 \cdot \epsilon_4 \Big) \tag{6.1.42}$$

where we have made use of:

$$\epsilon_i \cdot p_i = 0. \tag{6.1.43}$$

For Ward identity check we have:

$$\mathcal{A}_3^\mu p_{3\mu} = g_s^2 t^b_{i_1 i_2} f^{ba_3a_4}\frac{1}{s_{12}}\overline{u}_1\gamma^\mu v_2\Big(\epsilon_{4\mu}s_{12} - p_3\cdot\epsilon_4(p_3+p_4)_\mu\Big) \tag{6.1.44}$$

$$= g_s^2 t^b_{i_2 i_1} f^{ba_3a_4}\overline{u}_1\not\epsilon_4 v_2 \tag{6.1.45}$$

where we can define:

$$C_3 = t^b_{i_1 i_2} f^{ba_3a_4} \tag{6.1.46}$$

$$= -i[t^{a_3}, t^{a_4}]_{i_1 i_2} \tag{6.1.47}$$

$$= -i\Big(C_1 - C_2\Big) \tag{6.1.48}$$

which is now sufficient to show that $\mathcal{A}$ satisfies the Ward identity:

$$\mathcal{A}^\mu p_{3\mu} = g_s^2\overline{u}_1\not\epsilon_4 v_2\Big(- iC_1 + iC_2 - C_3\Big) = 0. \tag{6.1.49}$$

The color identity above can be used order to order the diagrams that appear in the amplitude. Defining:

$$\mathcal{A}_1 = g_s^2 C_1 A_1 \tag{6.1.50}$$

$$\mathcal{A}_2 = g_s^2 C_2 A_2 \tag{6.1.51}$$

$$\mathcal{A}_3 = g_s^2 i C_3 A_3 \tag{6.1.52}$$

we have:

$$\mathcal{A} = g_s^2\Big[C_1(A_1 + A_3) + C_2(A_1 - A_3)\Big] \tag{6.1.53}$$

where the *partial amplitude* (coefficient of the basis color factors C_1 and C_2) are related by the symmetry:

$$(A_1 + A_3)\Big|_{3\leftrightarrow 4} = A_2 - A_3. \tag{6.1.54}$$

It is therefore sufficient to compute only 2 of the 3 diagrams and then permute the result. This is *color ordering.*

Compact representation for the helicity amplitudes can be found using the *spinor helicity formalism.* We can write:

$$A(1_q^{h_1}2_q^{h_2}3_g^{h_3}4_g^{h_4}) = A_1 + A_3\Big|_{u_1\to\frac{1}{2}(1+h_1\gamma_5)u_1,\dots} \tag{6.1.55}$$

The formalism is built arounf the Weyl representation for the Dirac equation from which we construct everything from 2-component helicity spinors. We recall the Weyl representation:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \tag{6.1.56}$$

where $\sigma^\mu = (\mathbb{1}, \vec{\sigma})$ and $\overline{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})$ with $\vec{\sigma}$ the Pauli matrices. The helicity projection operators are:

$$P_\pm = \frac{1 \pm \gamma_5}{2} = \frac{1}{2}\begin{pmatrix} 1 \pm 1 & 0 \\ 0 & 1 \mp 1 \end{pmatrix} \quad (6.1.57)$$

from which we form *helicity spinors*:

$$u_+ = P_+ u = \begin{pmatrix} |p\rangle \\ 0 \end{pmatrix} = v_- \quad (6.1.58)$$

$$u_- = P_- u = \begin{pmatrix} |p] \\ 0 \end{pmatrix} = v_+. \quad (6.1.59)$$

Wcplòicit solutions for the 2-component Weyl spinors $|p\rangle$ and $|p]$ can be found satisfying:

$$|p\rangle\,[p| = \sigma \cdot p \quad \longleftrightarrow \quad \frac{1}{2}\langle p\sigma^\mu p] = p^\mu \quad (6.1.60)$$

$$|p]\langle p| = \overline{\sigma} \cdot p \quad \longleftrightarrow \quad \frac{1}{2}[p\sigma^\mu p\rangle = p^\mu \quad (6.1.61)$$

where $\langle p| = \epsilon\,|p\rangle$ and $[p| = \epsilon|p]$. Further details: Mangano, Parke Physics department Dixon.

Let's see some remarks to keep in mind. Helicity amplitudes don't interfore when squaring amplitudes:

$$\langle |\mathcal{A}|^2 \rangle = \frac{1}{4}\sum_n |\mathcal{A}(n)|^2 \quad , \quad 2 \to N. \quad (6.1.62)$$

Amplitudes are complex numbers, all the spinor indices contracted, and can be written in terms of *spinor products*:

$$\langle pq\rangle = -\langle qp\rangle \quad , \quad [pq] = -[qp] \quad (6.1.63)$$

where:

$$\langle pq\rangle[qp] = \delta_{pq} = (p+q)^2 = 2p \cdot q. \quad (6.1.64)$$

Weyl spinors ($|p\rangle$, $|p]$) provide a complete representation of the Lorentz group $SO(1,3)$ for massless objects:

$$SO(1,3) \overset{p^2=0}{\longleftrightarrow} SU(2) \otimes SU(2). \quad (6.1.65)$$

Polarization vectors can be defined with respect to a reference direction n^μ satisfying:

$$\epsilon \cdot p = 0 \quad , \qquad \epsilon \cdot n = 0 \quad (6.1.66)$$

$$\epsilon_+ \cdot \epsilon_- = -1 \quad , \qquad \epsilon_\pm \cdot \epsilon_\pm = 0 \quad (6.1.67)$$

$$\epsilon_+^\mu(p,n) = \frac{\langle n\sigma^\mu p]}{\sqrt{2}\langle np\rangle} \quad , \qquad \epsilon_-^\mu(p,n) = \frac{\langle p\sigma^\mu n]}{\sqrt{2}[np]}. \quad (6.1.68)$$

Exercise 7. Show the spin sum for the polarisation vectors is:

$$\sum_n \epsilon_n^\mu(p,n)\epsilon_n^\nu(p,n) = -g^{\mu\nu} + \frac{p^\mu n^\nu + p^\nu n^\mu}{p\cdot n} \tag{6.1.69}$$

which matces the propagator numerator in the light-like axial gauge:

$$\mathcal{L}_{gf} = \frac{1}{2\xi}(n\cdot A)^2. \tag{6.1.70}$$

We have also the *spinor product relations*:

$$\langle 12\rangle\langle 3| + \langle 23\rangle\langle 1| + \langle 31\rangle\langle 2| = 0 \tag{6.1.71}$$

$$\langle 12\rangle = \sqrt{|s_{12}|}e^{i\theta_{12}} \tag{6.1.72}$$

$$[12] = -\sqrt{|s_{12}|}e^{-i\theta_{12}} \tag{6.1.73}$$

$$\langle 1\sigma^\mu 2]\langle 3\sigma_\mu 4] = -2\langle 13\rangle[24] \tag{6.1.74}$$

where (6.1.71) is called *Schanten equation* and (6.1.74) is called *Fierz equation*. We are now ready to finish the partial amplitude computation for $q\bar{q}gg$. We have:

$$A_1 = -i\bar{u}\not{\epsilon}_4\frac{\not{p}_{23}}{s_{23}}\not{\epsilon}_3 v_2 \tag{6.1.75}$$

$$A_3 = -i\bar{u}_1\gamma^\mu v_2\frac{1}{s_{12}}\Big(\epsilon_3\cdot\epsilon_4(p_3-p_4)_\mu + 2p_3\cdot\epsilon_3\epsilon_{4\mu} - 2p_3\cdot\epsilon_4\epsilon_{3\mu}\Big) \tag{6.1.76}$$

fermion currents (with incoming convention) must run from + to − and we can use parity symmetry to restrict 1^-2^+ as the only independen, non-zero, choice.

We can also talk about *helicity* $-+++$. With a bit of spinor algebra we find:

$$A_1 = \frac{2i\langle 1n_4\rangle[41]\langle 1n_3\rangle[32]}{s_{23}\langle n_3 3\rangle\langle n_4 4\rangle} \tag{6.1.77}$$

$$A_3 = \frac{-i\langle 1\sigma^\mu 2]}{\langle n_4 4\rangle\langle n_3 3\rangle s_{12}}\Big(-2\langle n_3 n_4\rangle[34](p_3-p_4)_\mu + \langle n_4\sigma_\mu 4]\langle n_3\not{4}3] - \langle n_3\sigma_\mu 3]\langle n_4\not{3}4]\Big) \tag{6.1.78}$$

$$= \frac{-2i\langle 1n\rangle\Big(\langle n3\rangle[23] + \langle n4\rangle[24]\Big)[34]}{\delta_{12}\langle n_3 3\rangle\langle n_4 4\rangle} \tag{6.1.79}$$

where in the last step we fix $n_3 = n_4 = n$. So clearly for $n_3 = n_4 = 1$ all contributions vanish. We can see that *gauge invariance* implies Ward identity, which is biunivocal related with independency of ref. vectors. So:

$$A(1_{\bar{q}}^- 2_q^+ 3_g^+ 4_g^+) = 0. \tag{6.1.80}$$

Exercise 8. Show that:

$$A(1_{\bar{q}}^{-}2_{q}^{+}3_{g}^{+}4_{g}^{+}) = \frac{-2i\langle 31\rangle[24]^2}{\langle 12\rangle[12][23]} \tag{6.1.81}$$

$$= \frac{2i[24]^3[14]}{[12][23][34][41]} \tag{6.1.82}$$

$$= \frac{2i\langle 13\rangle^3\langle 23\rangle}{\langle 12\rangle\langle 23\rangle\langle 31\rangle\langle 41\rangle}. \tag{6.1.83}$$

6.1.3 Divergent graphs at 1-loop

The UV divergences are similar to those in QED. You can see the table 6.1.

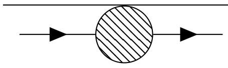
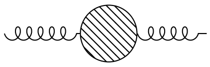
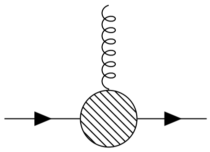
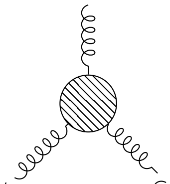
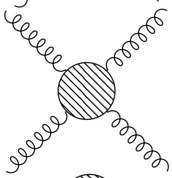
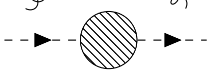
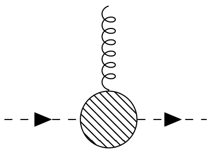
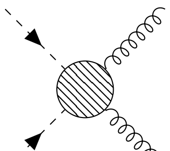
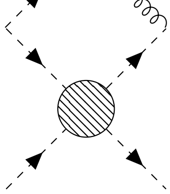
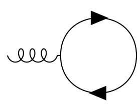
Γ	n_q	n_g	n_c	$\omega(\Gamma_{QED})$	name
	2	0	0	1	quark self-energy
	0	2	0	2	gluon self-energy
	2	1	0	0	$q\bar{q}g$-vertex
	0	3	0	1	$3g$-vertex
	0	4	0	0	$4g$-vertex
	0	0	2	2	ghost self-energy
	0	1	2	1	$c\bar{c}g$-vertex
	0	2	2	0	-
	0	0	4	0	-

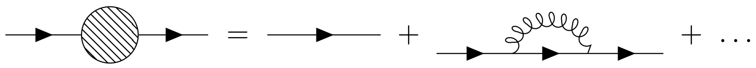
Table 6.1: Divergent diagrams of QCD.

In the table we have ommitted the tadpole graph:

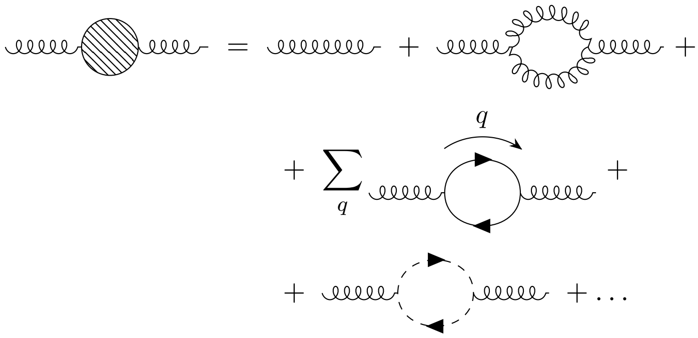


which is trivially zero (by color algebra this time). The last two graphs do not match gauge invariant operators and must therefore give *finite results* when all diagrams are summed together.

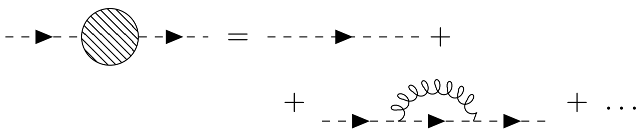
The major difference with respect to QED is the number of diagrams that contribute. Up to one-loop we have:



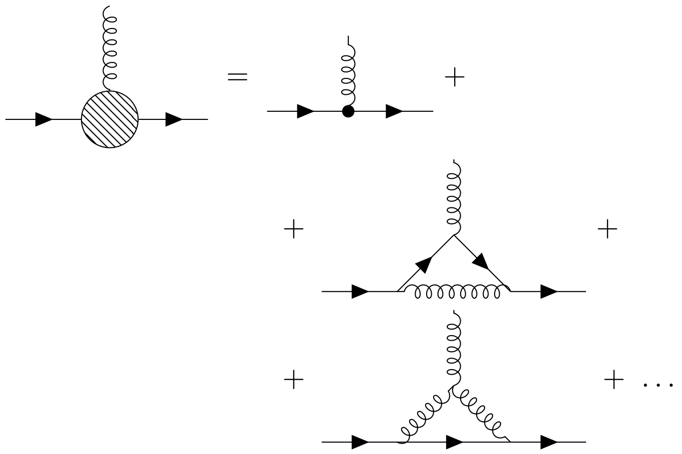
(6.1.84)



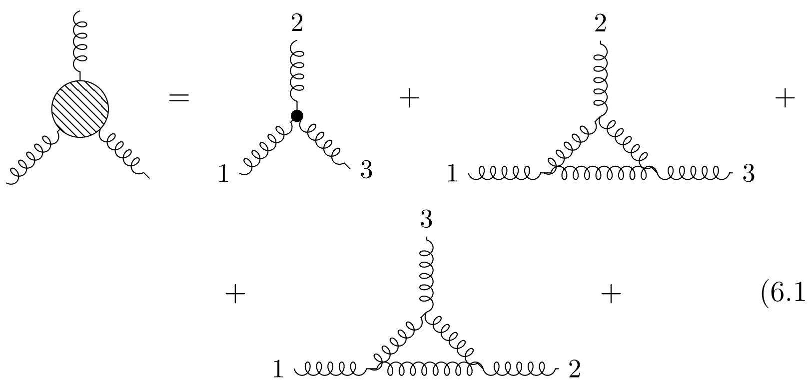
(6.1.85)



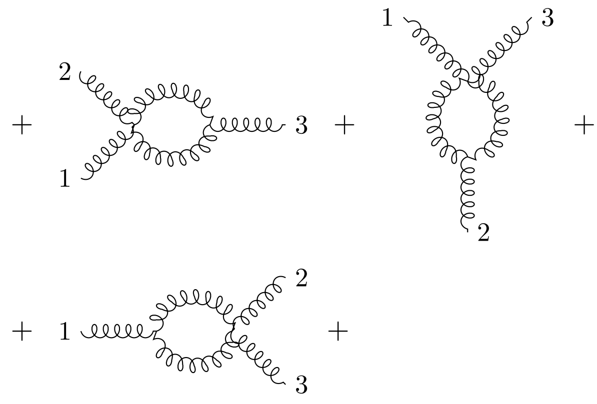
(6.1.86)



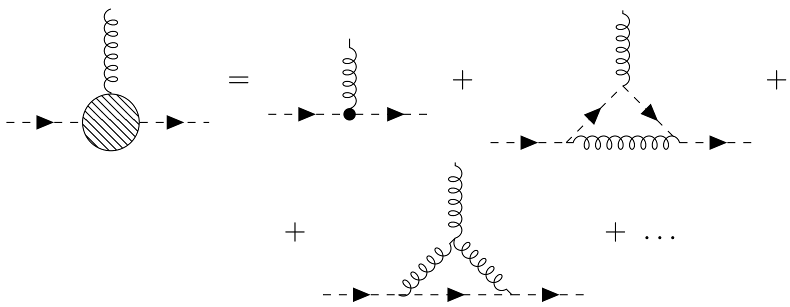
(6.1.87)



(6.1.88)



$$+ \text{ghost loops} + \text{quark loops} + \text{(higher order)} \quad (6.1.89)$$



(6.1.90)

4−point diagram expansions can be obtained relatively easily if required, clearly there are quite a few diagrams to consider.

6.1.4 Renormalization counter-terms

The necessary field and parameter redefinitions are as follows:

$$\psi_{q0} = Z_{\psi_q}^{1/2} \psi_{qR} \quad , \qquad A_0^\mu = Z_A^{1/2} A_R^\mu \quad (6.1.91)$$

$$c_0 = Z_c^{1/2} c_R \quad , \qquad m_{q0} = Z_{m_q} m_{qR} \quad (6.1.92)$$

$$g_{s0} = g_{sR} \mu_R^\epsilon Z_{gs}. \quad (6.1.93)$$

For the rest of this chapter quantities without R subscripts will be considered to be renormalized, e.g. $\psi_q = \psi_{qR}$. The QCD lagrangian in terms of renormalized quantities has the following counter-terms:

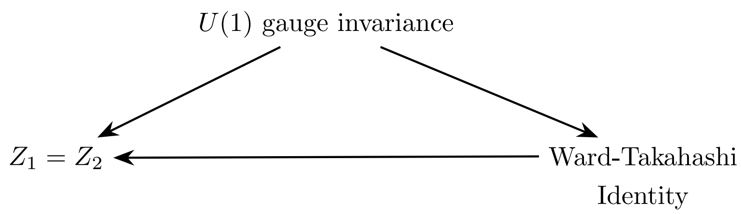
$$
\begin{aligned}
\mathcal{L}_{QCD} = \mathcal{L}_{QCD,R} + \\
& + \sum_q \Big[i\overline{\psi}_q \partial\!\!\!/ \psi_q \big(Z_{\psi_q} - 1\big) + m_q \overline{\psi}_q \psi_q \big(Z_{m_q} Z_{\psi_q} - 1\big) - \\
& - g_s \mu_R^{\epsilon} \overline{\psi}_q A\!\!\!/^a t^a \psi_q \big(Z_1 - 1\big) - \\
& - \overline{c}^a \Box c^a \big(Z_c - 1\big) - g_s \mu_R^{\epsilon} f^{abc} \partial_\mu \overline{c}^a A^{\mu b} c^c \big(Z_1^c - 1\big) + \\
& + \frac{1}{2} A_\mu^a \big(g^{\mu\nu} \Box - \partial^\mu \partial^\nu\big) A_\nu^a \big(Z_A - 1\big) + \\
& + g_s \mu_R^{\epsilon} f^{abc} A^{\mu b} A^{\nu c} \partial_\mu A_\nu^a \big(Z_1^{3g} - 1\big) - \\
& - \frac{1}{4} g_s^2 \mu_R^{2\epsilon} f^{abx} f^{xcd} A_\mu^a A_\nu^b A^{\mu c} A^{\nu d} \big(Z_1^{4g} - 1\big) \Big]
\end{aligned}
\qquad (6.1.94)
$$

where:

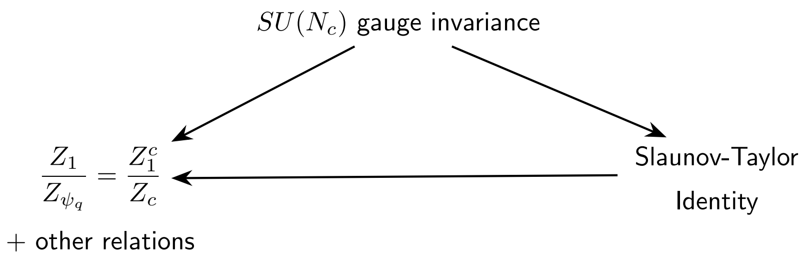
$$
Z_1 = Z_{g_s} Z_A^{1/2} Z_{\psi_q} \quad , \qquad Z_1^c = Z_{g_s} Z_A^{1/2} Z_c \qquad (6.1.95)
$$

$$
Z_1^{3g} = Z_{g_s} Z_A^{3/2} \quad , \qquad Z_1^{4g} = Z_{g_s}^2 Z_A^2. \qquad (6.1.96)
$$

Unlike QED the gauge invariance constraint on Z_x do not imply $Z_1 = Z_{\psi_q}$. There are a set of relations for correlation functions the follow form gauge invariance called *Slavrov-Taylor identities*, in turn we may derive additional constraints on Z_x but we cannot avoid the vertex amputation. So we have the scheme in figure 6.1.



(a) Scheme for QED.



(b) Scheme for QCD.

Figure 6.1

6.1.5 Strong CP problem

We finish this section with a small remark on the form of the QCD lagrangian. If we were to write down all $SU(N_c)$ guage invariant operators in 4 dimensions we would include:

$$F^{\mu\nu a}\tilde{F}^a_{\mu\nu} \tag{6.1.97}$$

where:

$$\tilde{F}^a_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma a}. \tag{6.1.98}$$

This term represents a subtle problem in QCD in which it's presence implies CP violating strong interaction which are not observed experimental.

Exercise 9. Show that:

$$F^{\mu\nu a}\tilde{F}^a_{\mu\nu} = \partial_\mu k^\mu \tag{6.1.99}$$

$$k^\mu = \epsilon^{\mu\nu\rho\sigma}A^a_\nu\left(F^a_{\rho\sigma} - \frac{g_s}{3}f^{abc}A^b_\rho A^c_\sigma\right). \tag{6.1.100}$$

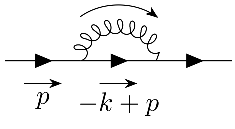
As a result we see the operator is a *total derivative* and plays no role in perturbation theory.

QED (abelian gauge theory) could also have a term $F^{\mu\nu}\tilde{F}_{\mu\nu}$ but this total derivative has no non-perturbative conseguences. 't Hooft showed that the QCD (non-abelian gauge theory) vacuum has *istanton* contributions, so $F\tilde{F}$ does have non-perturbative conseguences. A proposed solution to this problem introduces a new dynamical field, the *axion*, whic can naturally tune the coefficient of $F\tilde{F}$ to be very small (undetectable). This is the *Peccei-Quinn mechanism*. So far the axions have not been confirmed in experiment either.

6.2 One-loop graph computations

6.2.1 Quark self-energy

There is only one diagram and the only difference with respect to QED is the color factor:



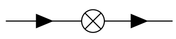
$$= t^a_{ik}t^a_{kj}g_s^2\mu_R^{2\epsilon}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\left(\frac{\gamma^\mu(-\not{k}+\not{p}+m_q)\gamma_\mu}{D(k,0)D(k-p,m_q)}\right) \tag{6.2.1}$$

$$= C_F\delta_{ij}\left(\frac{\alpha_s}{4\pi}\right)iC_\Gamma\left(\frac{1}{\epsilon}(\not{p}-4m_q)+\mathcal{O}(\epsilon^0)\right) \tag{6.2.2}$$

where we use:

$$\alpha_s = 4\pi g_s^2 \quad , \quad C_\Gamma = (4\pi)^\epsilon \Gamma(1+\epsilon). \tag{6.2.3}$$

We also have:



$$= i\delta_{ij}\Big(\delta_{\psi_q}\not{p} - \delta_3 m_q\Big) \tag{6.2.4}$$

$$= i\delta_{ij}\Big(\delta_{\psi_q}(\not{p} - m) - \delta_{m_q} m_q\Big). \tag{6.2.5}$$

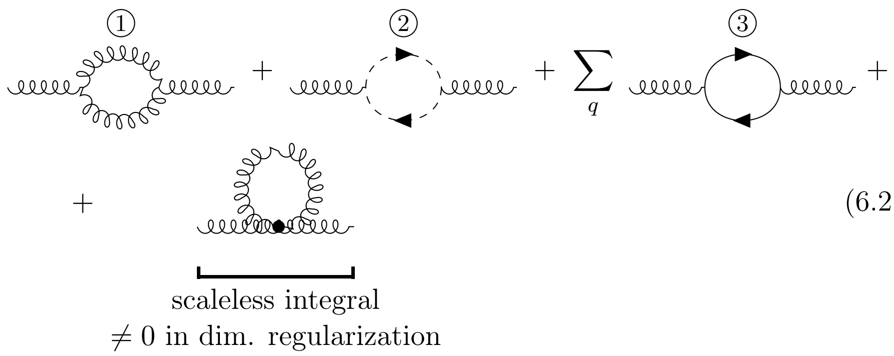
Since for the strong interaction there is no perturbative low energy limit, we cannot perform an on-shell renormalization and so stick to $\overline{MS}$:

$$\delta_{\psi_q}^{(1)} = \left(\frac{\alpha_s}{4\pi}\right) C_\Gamma \left(-\frac{1}{\epsilon}\right) \tag{6.2.6}$$

$$\delta_{m_q}^{(1)} = \left(\frac{\alpha_s}{4\pi}\right) C_\Gamma \left(-\frac{3}{\epsilon}\right). \tag{6.2.7}$$

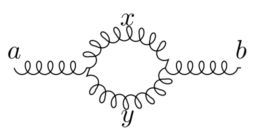
6.2.2 Gluon self-energy

The QCD computation differs from the QED analogue in that there are more diagrams and the ghost diagrams are required to produce the correct transverse tensor structure dictated by gauge invariance. We are saying:



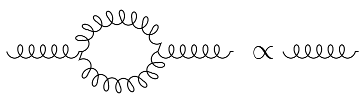
(6.2.8)

Let's start with the color part of the gluon diagram:



$$= f^{axy} f^{byx}. \tag{6.2.9}$$

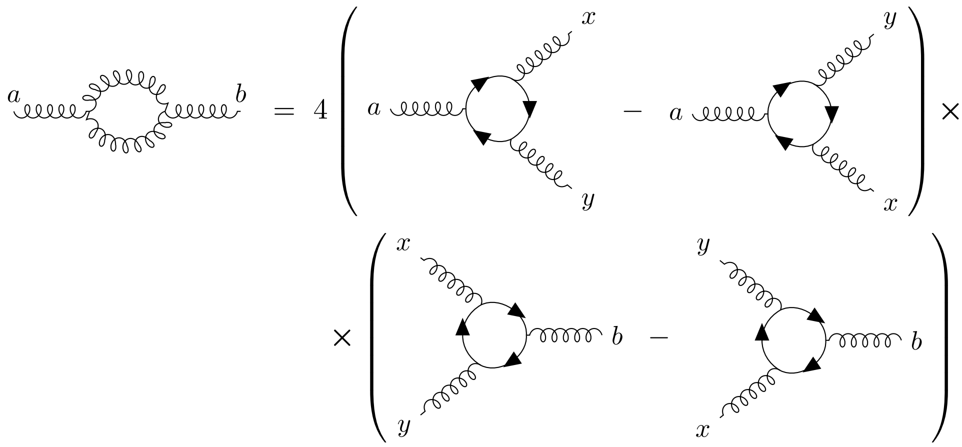
We may already anticipate that this could be proportional to the tree-level color factor δ^{ab}:



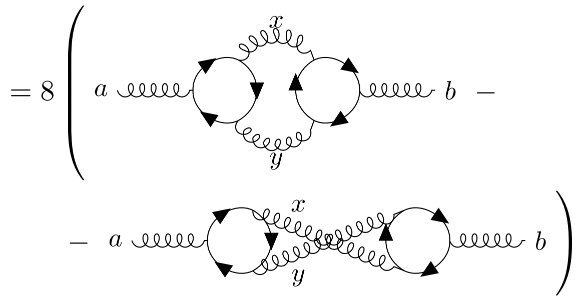
By writing:

$$f^{abc} = -2i\,\mathrm{Tr}\Big\{\Big[t^a, t^b\Big] t^c\Big\} \tag{6.2.10}$$

and applying the Fierz identity we can derive the constant of proportionality:

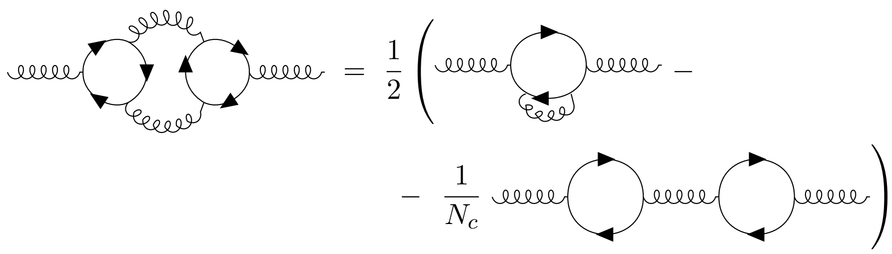


(6.2.11)

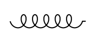


(6.2.12)

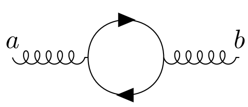
the first term is:



(6.2.13)

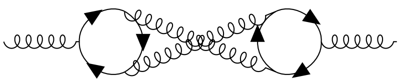
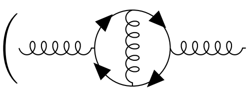
$$= \frac{1}{4}\left(C_F - \frac{1}{2N_c}\right)$$  (6.2.14)

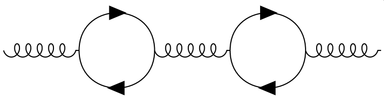
where:

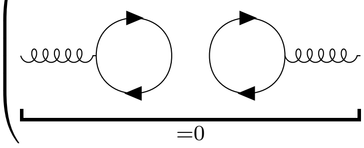


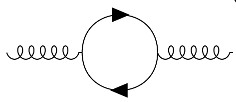
$$= \mathrm{Tr}\Big\{t^a t^b\Big\} = \frac{1}{2}\delta^{ab}. \tag{6.2.15}$$

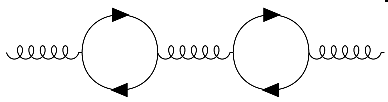
While the second term is:

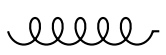
 $= \frac{1}{2}\Big($  $-$

$- \frac{1}{N_c}$  $\Big)$ (6.2.16)

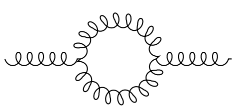
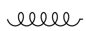
$= \frac{1}{2}\Big[\frac{1}{2}\Big($  $-$

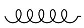
$- \frac{1}{N_c}$  $\Big) -$

$- \frac{1}{N_c}$  $\Big]$ (6.2.17)

$= -\frac{1}{4N_c}$  (6.2.18)

So we determine:

 $= 8\left(\frac{1}{4}\frac{N_c^2-1}{2N_c} - \frac{1}{8N_c} - \frac{1}{4N_c}\right)$  (6.2.19)

$= -N_c$  (6.2.20)

we identify the adjoint Casimir:

$$C_A = N_c. \tag{6.2.21}$$

Exercise 10. We can prove this relation using an alternative method which introduces the symmetric structure constants d^{abc}. Those are the steps:

- Using the definition:

$$\left\{t^a, t^b\right\} = d^{abc}t^c + \frac{1}{N_c}\delta^{ab} \tag{6.2.22}$$

  show:

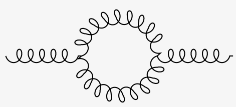
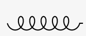
$$d^{abc} = 2\,\mathrm{Tr}\left\{\left\{t^a, t^b\right\}t^c\right\}. \tag{6.2.23}$$

- Multiply the relation:

$$\frac{1}{4}\left(d^{axy} + if^{axy}\right) = \mathrm{Tr}\{t^a t^x t^y\} \tag{6.2.24}$$

  with f^{byx} to find a relation for $f^{axy}f^{byx}$ in terms of traces. **NB:** $d^{axy}f^{byx} = 0$.

- Compute traces to arrive at:

 $= -C_A$  (6.2.25)

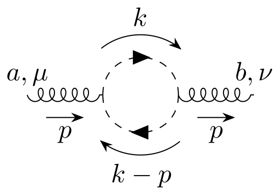
The color factor for the other diagrams are now clear as well (the subscript $1, 2, 3$ are related to the number in (6.2.8)):

$$d_1 = -C_A K_1 \delta_{ab} \tag{6.2.26}$$
$$d_2 = -C_A K_2 \delta_{ab} \tag{6.2.27}$$
$$d_3 = \frac{1}{2} K_3 \delta_{ab}. \tag{6.2.28}$$

The kinematic factor K_3 is identical to the photon self-energy, so we may immediately arrive at:

$$K_3 = i\left(\frac{\alpha_s}{4\pi}\right) C_\Gamma \Big(- g^{\mu\nu}p^2 + p^\mu p^\nu\Big) \left(-\frac{4}{3\epsilon} + \mathcal{O}(\epsilon^0)\right). \tag{6.2.29}$$

The ghost loop is perhaps the next simplest:



$$= -C_A\delta_{ab}\left(i^2 g_s^2 \mu_R^{2\epsilon}\right)\int_k \frac{k^\mu (k-p)^\nu}{D(k,0)D(k-p,0)} \tag{6.2.30}$$

$$= -C_A\delta_{ab}K_2. \tag{6.2.31}$$

Exercise 11. Show:

$$K_2 = -g_s^2\mu_R^{2\epsilon}\left(p^2 g^{\mu\nu} + (d-2)p^\mu p^\nu\right)\frac{1}{4(d-1)} I_{2,00}^{[4-2\epsilon]}[1] \tag{6.2.32}$$

$I_{2,00}^{[4-2\epsilon]}[1]$ massless bubble integral computed in Chapter §3.

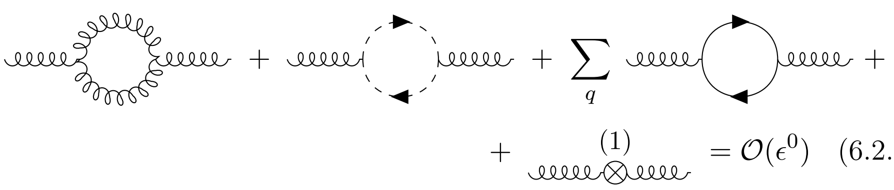
Exercise 12. Show the gluon diagram results in:

$$K_3 = -\frac{1}{2}g_s^2\mu_R^{2\epsilon}\Big[-(6d-5)p^2 g^{\mu\nu} + (7d-6)p^\mu p^\nu\Big]\times \\ \times \frac{1}{2(d-1)} I_{2,00}^{[4-2\epsilon]}[1]. \tag{6.2.33}$$

Exercise 13. Show that the sum of all diagrams gives:

$$d_1 + d_2 + \sum_q d_3 = \left(\frac{\alpha_s}{4\pi}\right) C_\Gamma \Big(-p^2 g^{\mu\nu} + p^\mu p^\nu\Big)\times \\ \times \frac{1}{\epsilon}\left(\frac{5}{3}C_A - \frac{2}{3}n_f\right) + \mathcal{O}(\epsilon^0). \tag{6.2.34}$$

Combining with te counterterm we determine $\delta_A^{(1)}$:



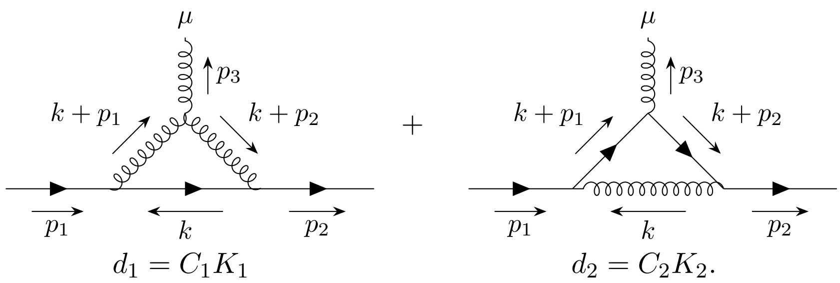
$$= \mathcal{O}(\epsilon^0) \tag{6.2.35}$$

using $\overline{MS}$, so we get:

$$\delta_A^{(1)} = \left(\frac{\alpha_s}{4\pi}\right) C_\Gamma \left(\frac{5C_A}{3} - \frac{2n_f}{3}\right)\frac{1}{\epsilon}. \tag{6.2.36}$$

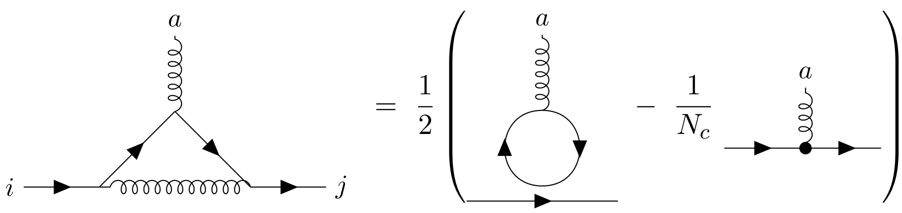
6.2.3 The quark-gluon vertex

The two contributing one-loop diagrams are:



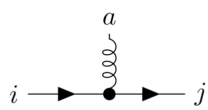
(6.2.37)

The color factors are simple to determine:



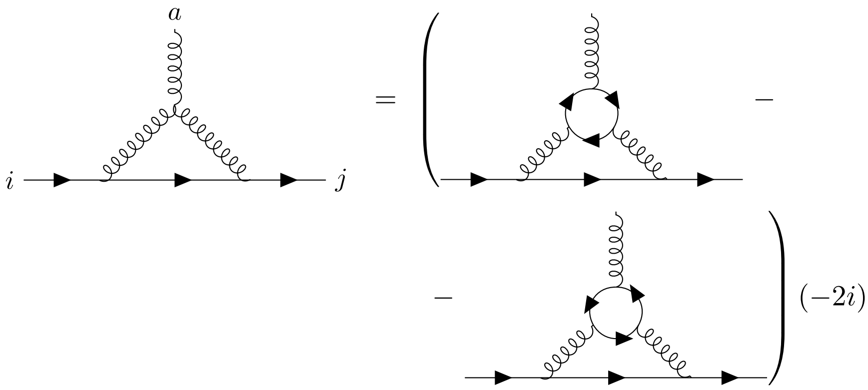
(6.2.38)

$$= -\frac{1}{2N_c}$$



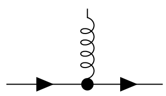
(6.2.39)

$$= C_2 \qquad (6.2.40)$$



(6.2.41)

$$= -i\frac{N_c}{2}$$



(6.2.42)

$$= C_1 \qquad (6.2.43)$$

where the step (6.2.42) is left as exercise.

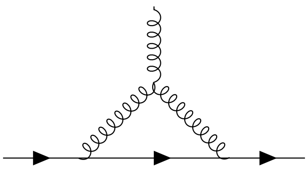
Exercise 14. Show the kinematic contributions are:

$$K_1 = \left(\frac{\alpha_s}{4\pi}\right) C_\Gamma i g_s \overline{u}_2 \gamma^\mu v_1 \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0) \tag{6.2.44}$$

$$K_2 = \left(\frac{\alpha_s}{4\pi}\right) C_\Gamma i g_s \overline{u}_2 \gamma^\mu v_1 \left(\frac{3i}{\epsilon}\right) + \mathcal{O}(\epsilon^0) \tag{6.2.45}$$

where the IR divergences in the scalar integrals $I^{[4-2\epsilon]}_{3,0mm}$ and $I^{[4-2\epsilon]}_{3,000}$ have been discorded.

Combining with the contribution from the vertex counter-term we find ($\overline{MS}$):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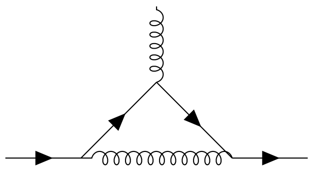
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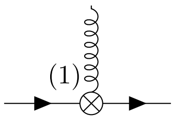


$= \mathcal{O}(\epsilon^0)$ (6.2.46)

so:

$$\delta_1^{(1)} = \left(\frac{\alpha_s}{4\pi}\right) C_\Gamma \left(-\frac{3N_c}{2} + \frac{1}{2N_c}\right)\left(\frac{1}{\epsilon}\right) \tag{6.2.47}$$

$$= \left(\frac{\alpha_s}{4\pi}\right) C_\Gamma \left(C_A + C_F\right)\left(-\frac{1}{\epsilon}\right). \tag{6.2.48}$$

6.3 Renormalization of the strong coupling

From the relation:

$$Z_1 = Z_{g_s} Z_A^{1/2} Z_{\psi_q} \tag{6.3.1}$$

we obtain:

$$\delta_1^{(1)} - \delta_{\psi_q}^{(1)} - \frac{1}{2}\delta_A^{(1)} = \delta_{g_s}^{(1)} \tag{6.3.2}$$

hence:

$$\delta_{g_s}^{(1)} = \left(\frac{\alpha_s}{4\pi}\right) C_\Gamma \left[-\frac{1}{\epsilon}(C_A + C_F) - \left(-\frac{1}{\epsilon}\right) - \frac{1}{2}(5C_A - 2n_f)\frac{1}{3\epsilon}\right] \tag{6.3.3}$$

$$= -\left(\frac{\alpha_s}{4\pi}\right) C_\Gamma \left(\frac{11C_A - 2n_f}{6\epsilon}\right) \tag{6.3.4}$$

to compare this directly with QED ($U(1)$) we can identify the quark loop with a factor of T_R, where $T_R = 1/2$ for QCD and $T_R = 1$ for QED, so:

$$\delta_{g_s}^{(1)} = -\left(\frac{\alpha_s}{4\pi}\right) C_\Gamma \left(\frac{11C_A - 4T_R n_f}{6\epsilon}\right) \tag{6.3.5}$$

$$\delta_e^{(1)} = \delta g_s^{(1)}\Big|_{\substack{C_A\to 0\\ T_R n_f \to 1\\ \alpha_s\to\alpha}} \tag{6.3.6}$$

$$= \left(\frac{\alpha_s}{4\pi}\right) C_\Gamma \frac{2}{3\epsilon}. \tag{6.3.7}$$

The charge screening effect seen in QED implied that:

$$\alpha_{eff}(Q^2) > \alpha_{eff}(0) \tag{6.3.8}$$

for a scale $Q^2 > 0$. The analogous effect in QCD will depend on the sign of δ_{g_s}. For $N_c = 3$ there is a sign change (with respect to QED) if:

$$33 - 2n_f > 0 \tag{6.3.9}$$

hence:

$$n_f \leq 16 \tag{6.3.10}$$

remembering that n_f must be integer. The Standard Model has $n_f = 6$ and so:

$$\alpha_{s,eff}(Q^2) < \alpha_{s,eff}(Q'^2) \tag{6.3.11}$$

for:

$$Q^2 > Q'^2. \tag{6.3.12}$$

Chapter 7

The renormalization group

After a brief historical rewiew, we showed hot the renormalization procedure couses model parameters to evolve with energy scale. We saw how the β function describes how the coupling runs with energy. Also, we saw how the Callan-Symanzik equation follows from the fact the renormalization scale dependence is a conseguence of absorbing UV divergences.

7.1 History

After the renormalization of QED, the systematics of the procedure and conseguences on physical observables were still not fully understood.

Between 1950-1970 there were several studies of how residual scale dependence on *renormalized quantities* in perturbation theory implied *evolution equations* for model parameters (coupling, masses etc.). The first works were by *Stuekelberg, Peterman* (1953) and *Gell-Mann, Low* (1954) studying electric charge in QED. Same ideas came from Condensed matter systems (*Kadanoff*, 1966). The ideas were formalized in the works of *Callan-Symanzik* (1970) and *Wilson* (1971-1975).

There were several Nobel prize for these works: Feynman, Schwinger, Tomonaga (1965), Gell-Mann (1969), Glashow, Salam, Weinberg (1979), Wilson (1982), 't Hooft, Veltman (1999), Gross, Politter, Wilczek (2004).

7.1.1 Scale dependence of electric charge

Let's see the work by Gell-Mann and Low. The reference for this section is Itzykson-Zuber [4] at page 633-635.

We can use our result for $\Pi^{\mu\nu}$ (5.2.71) at one-loop to study the effective charge at different scales. Recall also (5.2.74):

$$\tilde{G}_{\mu\nu}(q^2,\dots) = \frac{-ig^{\mu\nu}}{q^2}\frac{1}{1+\Pi(q^2,\dots)} + G_L^{\mu\nu}(q^2,\dots).$$

The renormalized vacuum polarization is a function of q^2, α and the mass m:

$$\Pi(\alpha, q^2, m^2) \tag{7.1.1}$$

and Π was renormalized on-shell through the condition:

$$\Pi(\alpha, 0, m^2) = 0. \tag{7.1.2}$$

We can define an effective charge at the scale q^2, called $d(\alpha, q^2, m^2)$, by:

$$\alpha\tilde{G}^{\mu\nu} = \frac{-ig^{\mu\nu}}{q^2} d(\alpha, q^2, m^2) + \alpha\tilde{G}_L^{\mu\nu} \tag{7.1.3}$$

which implies:

$$d(\alpha, q^2, m^2) = \frac{\alpha}{1 + \Pi(\alpha, q^2, m^2)} \tag{7.1.4}$$

and at $q^2 = 0$:

$$d(\alpha, q^2, m^2) = \alpha. \tag{7.1.5}$$

What happens if we fix the charge at a different point, say at $q^2 = \lambda^2$, with $\lambda < 0$ (so space-like):

$$\alpha_\lambda = d(\alpha, \lambda^2, m^2) = \frac{\alpha}{1 + \Pi(\alpha, \lambda^2, m^2)} \tag{7.1.6}$$

we can now re-write α in terms of α_λ such that:

$$d(\alpha, q^2, m^2) = D(\alpha_\lambda, q^2, m^2; \lambda^2) \tag{7.1.7}$$

where the new expression for the effective charge satisfies:

$$\alpha_\lambda = D(\alpha_\lambda, \lambda^2, m^2; \lambda^2) \tag{7.1.8}$$

d, the effective charge, is dimensionless so we can rewrite (7.1.7), in region $q^2 < 0$, $m^2 > 0$ and $\lambda^2 < 0$, as:

$$d(\alpha, -q^2/m^2) = D(\alpha_\lambda, q^2/\lambda^2; -\lambda^2/m^2) \tag{7.1.9}$$

in which:

$$\alpha_\lambda = d(\alpha, -\lambda^2/m^2) = D(\alpha_\lambda, 1, -\lambda^2/m^2). \tag{7.1.10}$$

We now study (7.1.9) in the asymptotic limit $-q^2/m^2 \to \infty$:

$$\Pi(\alpha, -q^2/m^2) \overset{-q^2/m^2 \to \infty}{\longrightarrow} \Pi^{asy}(\alpha, -q^2/m^2). \tag{7.1.11}$$

We computed this directly in §5 (see (5.4.7)) in dimensional regularization:

$$\Pi^{asy}_{\overline{MS}}(\alpha, -q^2/m^2) = \left(\frac{\alpha}{4\pi}\right) C_\Gamma \frac{4}{3}\left(-\frac{5}{3} + 3\log\left(\frac{q^2}{m^2}\right)\right) + \mathcal{O}\left(\alpha^2\right)$$

so we can show that:

$$D(\alpha_\lambda, q^2/\lambda^2; -\lambda^2/m^2) \overset{-q^2/m^2\to\infty}{\longrightarrow} D^{asy}\left(\alpha_\lambda, \left(-q^2/m^2\right)\left(-m^2/\lambda^2\right)\right) \tag{7.1.12}$$

$$d\left(\alpha, -q^2/m^2\right) \overset{-q^2/m^2\to\infty}{\longrightarrow} d^{asy}\left(\alpha, -q^2/m^2\right) \tag{7.1.13}$$

so that the asymtotic limit of (7.1.9) is:

$$d^{asy}(\alpha, x) = D^{asy}(\alpha_y, x/y) \tag{7.1.14}$$

where:

$$x = -\frac{q^2}{m^2} \quad , \quad y = -\frac{\lambda^2}{m^2} \tag{7.1.15}$$

and:

$$\alpha_y = d^{asy}(\alpha, y) = D^{asy}(\alpha_y, 1). \tag{7.1.16}$$

The *Gell-Mann, Low function* is defined as:

$$\psi(z) = \frac{\partial}{\partial x} D^{asy}(z, x)\bigg|_{x=1}. \tag{7.1.17}$$

We can use (7.1.9) to prove:

$$\psi(\alpha_x) = x\frac{\partial}{\partial x} d^{asy}(\alpha, x). \tag{7.1.18}$$

Proof. Starting with:

$$\frac{\partial}{\partial x} d^{asy}(d, x) = \frac{\partial}{\partial x} D^{asy}(\alpha_y, x/y) \tag{7.1.19}$$

$$= \frac{\partial(x/y)}{\partial x}\frac{\partial}{\partial(x/y)} D^{asy}(\alpha_y, x/y) \tag{7.1.20}$$

$$= \frac{1}{y}\frac{\partial}{\partial z} D^{asy}(\alpha_y, z) \tag{7.1.21}$$

writing $z = x/y$. The LHS (Left-Hand-Side) of this equation is independent of y, so we may set $y = x$ to find:

$$\frac{\partial}{\partial x} d^{asy}(\alpha, x) = \frac{1}{x}\left(\frac{\partial}{\partial x} D^{asy}(\alpha_x, x)\right)_{x=1} \tag{7.1.22}$$

$$= \frac{1}{x}\psi(\alpha_x). \tag{7.1.23}$$

So the function $\psi(\alpha)$ determines the evolution of the effective charge (in the asymptotic region).

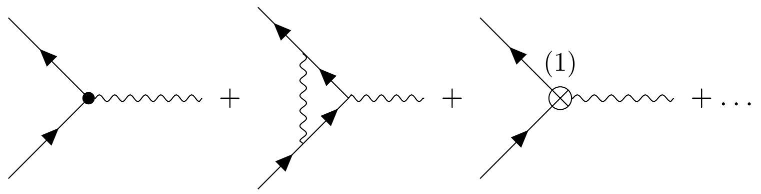
So, what we need to keep in mind is:

- $\tilde{G}^{\mu\nu}$ does not change, and it is independent of the renormalization scheme.
- The effective charge does change when we change the scale, at which we fixed the renormalization scheme. The scale dependence is given by the Gell-Mann Low function $\psi(x)$.
- Callan-Symanzik clarified and generalized this picture.

7.2 Scale dependence of perturbative predictions

Consider the perturbative corrections to the vertex function which can be related to a vertex function, which can be related to a physical observable O (in QED):

$O(q^2) \sim$



$$= \lambda O^{(1)}(q^2) + \lambda^3 O^{(2)}(q^2; \mu_R^2). \tag{7.2.1}$$

Let's compare this to an experiment at q_0^2, where we measure $O(q^2) = \sigma$ and so, at leading order (LO):

$$\sigma = \lambda \sigma^{(1)}(q_0^2) + \mathcal{O}(\lambda^3) \tag{7.2.2}$$

$$\implies \quad \lambda_{LO} = \frac{\sigma}{\sigma^{(1)}(q_0^2)}. \tag{7.2.3}$$

If we compare theory and experiment at next-to-leading-order (NLO):

$$\sigma = \lambda O^{(1)}(q_0^2) + \lambda^3 \sigma^{(2)}(q_0^2; \mu_R^2) + \mathcal{O}(\lambda^5). \tag{7.2.4}$$

The measurement has not changed, but the value of the coupling has:

$$\lambda_{LO} \sigma^{(1)}(q_0^2) = \lambda_{NLO} \sigma^{(1)}(q_0^2) + \lambda_{NLO}^3 \sigma^{(2)}(q_0^2; \mu_R^2). \tag{7.2.5}$$

For a single scale process, such as this one, the NLO correction will take the form:

$$\sigma^{(2)}(q_0^2; \mu_R^2) = \sigma^{(1)}(q_0^2) \left(a \log\left(\frac{\mu_R^2}{q_0^2}\right) + b \right) \tag{7.2.6}$$

and so:

$$\lambda_{LO} = \lambda_{NLO} + \lambda_{NLO}^3 \left(a \log\left(\frac{\mu_R^2}{q_0^2}\right) + b \right). \tag{7.2.7}$$

7.3 The Callan-Symanzik equation

At the end of Chapter §3 we considered the scale dependence of the renormalized correlation functions. Comparing bare and renormalized functions implied:

$$\tilde{G}_0(p_1,\ldots,p_n;\lambda_0) = FT\left(\langle 0|\, T\{\phi_0(x_1)\ldots\phi_0(x_n)\}\,|0\rangle\right) \tag{7.3.1}$$
$$= Z_\phi^{n/2}(\mu_R,\lambda_R(\mu_R))\tilde{G}_R(p_1,\ldots,p_n;\lambda_R(\mu_R),\mu_R). \tag{7.3.2}$$

From this expression we derive:

$$0 = \mu_R\frac{\mathrm{d}}{\mathrm{d}\mu_R}\tilde{G}_0 \tag{7.3.3}$$
$$= \mu_R\frac{\mathrm{d}}{\mathrm{d}\mu_R}\left(Z_\phi^{n/2}\tilde{G}_R\right) \tag{7.3.4}$$
$$= \mu_R\left(\frac{\mathrm{d}}{\mathrm{d}\mu_R}Z_\phi^{n/2}\right)\tilde{G}_R + \mu_R Z_\phi^{n/2}\left(\frac{\mathrm{d}}{\mathrm{d}\mu_R}\tilde{G}_R\right) \tag{7.3.5}$$
$$= n(Z_\phi^{1/2})^{n-1}\mu_R\left(\frac{\mathrm{d}}{\mathrm{d}\mu_R}Z_\phi^{1/2}\right)\tilde{G}_R + Z_\phi^{n/2}\mu_R\frac{\mathrm{d}}{\mathrm{d}\mu_R}\tilde{G}_R \tag{7.3.6}$$
$$= Z_\phi^{n/2}\left(\mu_R\frac{\mathrm{d}}{\mathrm{d}\mu_R}\tilde{G}_R + n\tilde{G}_R\frac{\mathrm{d}}{\mathrm{d}\mu_R}\log\left(Z_\phi^{1/2}\right)\right). \tag{7.3.7}$$

For a *minimal scheme* (so MS and $\overline{MS}$ etc.) we may state:

$$Z_\phi \equiv Z_\phi(\lambda_R(\mu_R)) \tag{7.3.8}$$

and so the relation becomes:

$$\mu_R\frac{\partial}{\partial\mu_R}\tilde{G}_R(\mu_R,\lambda(\mu_R)) + \mu_R\frac{\mathrm{d}\lambda(\mu_R)}{\mathrm{d}\mu_R}\frac{\partial}{\partial\lambda}\tilde{G}_R(\mu_R,\lambda) + \frac{n}{2}\mu_R\left(\frac{\mathrm{d}\log Z_\phi}{\mathrm{d}\mu_R}\right)\tilde{G}_R(\mu_R,\lambda(\mu_R)) = 0. \tag{7.3.9}$$

The scale-dependence of λ and Z_ϕ appear through what we call **beta function**:

$$\beta(\lambda) = \mu_R\frac{\mathrm{d}\lambda(\mu_R)}{\mathrm{d}\mu_R} \tag{7.3.10}$$

and the **anomalous dimension**:

$$\gamma_\phi(\lambda) = \frac{1}{2}\mu_R\frac{\mathrm{d}\log Z_\phi}{\mathrm{d}\mu_R} \tag{7.3.11}$$

that can be written as:

$$\gamma_\phi(\lambda) = \frac{1}{2}\mu_R\frac{\mathrm{d}\log Z_\phi}{\mathrm{d}\mu_R} \tag{7.3.12}$$
$$= \frac{1}{2}\mu_R\frac{\mathrm{d}\lambda}{\mathrm{d}\mu_R}\frac{\mathrm{d}}{\mathrm{d}\lambda}\log Z_\phi \tag{7.3.13}$$
$$= \frac{1}{2}\mu_R\beta(\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}\log Z_\phi. \tag{7.3.14}$$

We have arrived at the **Callan-Symanzik equation**:

$$\left(\mu_R\frac{\partial}{\partial\mu_R}+\beta(\lambda)\frac{\partial}{\partial\lambda}+n\gamma_\phi(\lambda)\right)\tilde{G}_R=0. \tag{7.3.15}$$

Note. This is the simplest case of massless scalar field theory. There will be additional terms for masses and other fields. For example, for *massive scalar theory* we have:

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}+\gamma_m(\lambda)m\frac{\partial}{\partial m}+n\gamma_\phi(\lambda)\right)\times \\ \times\,\tilde{G}(\mu,\lambda(\mu),m(\mu))=0. \tag{7.3.16}$$

For *QED* we have:

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(e)\frac{\partial}{\partial e}+\gamma_m(e)m\frac{\partial}{\partial m}+n\gamma_A(e)+\right. \\ \left.+2k\gamma_\psi(e)\right)\tilde{G}_{n,2k}(\mu,e(\mu),m(\mu))=0. \tag{7.3.17}$$

7.4 Extraction of β and γ from UV counterterms

For a *minimal scheme* there. is a direct connection between the perturbative expansion of conunterterms, $1-Z_x$, and β/γ. For example:

$$\lambda_0=\lambda\mu^\epsilon Z_\lambda \quad , \quad Z_x=1+\delta_\lambda. \tag{7.4.1}$$

The perturbation expansion of δ_λ is:

$$\delta_\lambda=\sum_{L=1}^{\infty}\lambda^{2L}\delta_\lambda^{(L)} \tag{7.4.2}$$

but $\delta_\lambda^{(L)}$ also has a series expansion in dimensional regularization:

$$\delta_\lambda^{(L)}=\sum_{k=-L}^{\infty}\epsilon^k\delta_\lambda^{L,k} \tag{7.4.3}$$

and in a minimal scheme the upper bound of the ϵ expansion will be -1 (only poles). Hence:

$$\delta_\lambda = \sum_{L=1}^{\infty} \sum_{k=-L}^{-1} \lambda^{2L} \epsilon^k \delta_\lambda^{(L,k)} \tag{7.4.4}$$

$$= \sum_{L=1}^{\infty} \sum_{k=1}^{L} \frac{\lambda^{2L}}{\epsilon^k} \delta_\lambda^{L,k} \tag{7.4.5}$$

re-label with the values for k

$$= \sum_{k=1}^{\infty} \sum_{L=k}^{\infty} \frac{\lambda^{2L}}{\epsilon^k} \delta_\lambda^{(L,k)} \tag{7.4.6}$$

change under of the summation

$$= \sum_{k=1}^{\infty} \frac{a_k(\lambda)}{\epsilon^k} \tag{7.4.7}$$

where we use:

$$a_k(\lambda) = \sum_{L=k}^{\infty} \lambda^{2L} \delta_\lambda^{(L,k)}. \tag{7.4.8}$$

Now, we can look at the ϵ and λ dependence of β:

$$\beta(\lambda, \epsilon) = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \lambda \tag{7.4.9}$$

$$= \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \left(\lambda_0 \mu^{-\epsilon} Z_\lambda^{-1} \right) \tag{7.4.10}$$

$$= -\epsilon \lambda_0 \mu^{-\epsilon} Z_\lambda^{-1} + \lambda_0 \mu^{-\epsilon} \mu \frac{\mathrm{d}Z_\lambda^{-1}}{\mathrm{d}\mu} \tag{7.4.11}$$

$$= -\epsilon\lambda + \lambda Z_\lambda \mu \frac{\mathrm{d}Z_\lambda^{-1}}{\mathrm{d}Z_\lambda} \frac{\mathrm{d}Z_\lambda}{\mathrm{d}\mu} \tag{7.4.12}$$

$$= -\epsilon\lambda - \lambda Z_\lambda^{-1} \beta(\lambda, \epsilon) \frac{\mathrm{d}Z_\lambda}{\mathrm{d}\lambda} \tag{7.4.13}$$

so we get:

$$\beta(\lambda, \epsilon) \left(Z_\lambda + \lambda \frac{\mathrm{d}Z_\lambda}{\mathrm{d}\lambda} \right) + \epsilon \lambda Z_\lambda = 0. \tag{7.4.14}$$

The ϵ expansion of β is:

$$\beta(\lambda, \epsilon) = \sum_{k=0}^{\infty} \epsilon^k \beta^{(k)}(\lambda) \tag{7.4.15}$$

and we are interested in terms up to $\mathcal{O}(\epsilon^0)$, so we can expand (7.4.14) up to order $N > 0$:

$$\begin{aligned} &\left(\beta^{(0)} + \epsilon\beta^{(1)} + \cdots + \epsilon^N \beta^{(N)}\right) \times \\ &\times \left(1 + \frac{a_1}{\epsilon} + \frac{a_2}{\epsilon} + \cdots + \right. && (7.4.16) \\ &\times \left. \frac{1}{\epsilon}\lambda\frac{\mathrm{d}a_1}{\mathrm{d}\lambda} + \frac{1}{\epsilon^2}\lambda\frac{\mathrm{d}a_2}{\mathrm{d}\lambda} + \dots \right) + && (7.4.17) \\ &+ \epsilon\lambda\left(1 + \frac{a_1}{\epsilon} + \frac{a_2}{\epsilon^2} + \dots\right) && (7.4.18) \\ &= 0 && (7.4.19) \end{aligned}$$

where in (7.4.16), (7.4.17) and (7.4.18) there are the expansion respectively of Z_λ, $\lambda\frac{\mathrm{d}Z_\lambda}{\mathrm{d}\lambda}$ and $\epsilon\lambda Z_\lambda$. The only term at $\mathcal{O}(\epsilon^N)$ is $\beta^{(N)}$ so:

$$\mathcal{O}(\epsilon^N) \implies \beta^{(N)}\epsilon^N \tag{7.4.20}$$

with $\beta^{(N)} = 0$ we have:

$$\mathcal{O}(\epsilon^{N-1}) \implies \beta^{(N-1)} = 0 \tag{7.4.21}$$

for $N > 1$. At $\mathcal{O}(\epsilon)$ we see the last contribution $\epsilon\lambda Z_\lambda$:

$$\begin{aligned} &\mathcal{O}(\epsilon^N) &&\implies \beta^{(N)} = 0 && (7.4.22) \\ &\mathcal{O}(\epsilon^{N-1}) &&\implies \beta^{(N-1)} = 0 && (7.4.23) \\ &\vdots \\ &\mathcal{O}(\epsilon) &&\implies \beta^{(1)} = -\lambda && (7.4.24) \\ &\mathcal{O}(\epsilon^0) &&\implies \beta^{(0)} = \lambda^2 \frac{\mathrm{d}}{\mathrm{d}\lambda} a_1 && (7.4.25) \\ &\mathcal{O}(\epsilon^{-1}) &&\implies \beta^{(0)}\frac{\mathrm{d}(\lambda a_1)}{\mathrm{d}\lambda} = \lambda^2 \frac{\mathrm{d}}{\mathrm{d}\lambda} a_2 && (7.4.26) \\ &\vdots \\ &\mathcal{O}(\epsilon^{-k}) &&\implies \beta^{(0)}\frac{\mathrm{d}(\lambda a_k)}{\mathrm{d}\lambda} = \lambda^2 \frac{\mathrm{d}}{\mathrm{d}\lambda} a_{k+1} && (7.4.27) \end{aligned}$$

where in (7.4.25) we use the fact:

$$\begin{aligned} 0 &= \epsilon\beta^{(1)}\frac{a_1}{\epsilon} + \beta^{(0)} + \epsilon\beta^{(1)}\frac{\lambda}{\epsilon}\frac{\mathrm{d}a_1}{\mathrm{d}\lambda} + \epsilon\lambda\frac{a_1}{\epsilon} && (7.4.28) \\ &\underset{\beta^{(1)}=-\lambda}{=} \beta^{(0)} - \lambda^2\frac{\mathrm{d}a_1}{\mathrm{d}\lambda} && (7.4.29) \end{aligned}$$

and in (7.4.26):

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda a_1) = \lambda\frac{\mathrm{d}a_1}{\mathrm{d}\lambda} + a_1. \tag{7.4.30}$$

So $\beta^{(0)} = \beta(\lambda, 0)$ is the beta function in the CS equation and appears in multiple relations, so we find relations between the poles of δ_λ, recalling (7.4.8), at different orders in ϵ and L:

$$\beta^{(0)} = \lambda^2 \frac{\mathrm{d}}{\mathrm{d}\lambda} a_1 \tag{7.4.31}$$

$$\beta^{(0)} \frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda a_1) = \lambda^2 \frac{\mathrm{d}}{\mathrm{d}\lambda} a_2 \tag{7.4.32}$$

$$\Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\lambda} a_1 \frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda a_1) = \frac{\mathrm{d}}{\mathrm{d}\lambda} a_2 \tag{7.4.33}$$

$$\overset{\mathcal{O}(\lambda^3)}{\Longrightarrow} \quad \left(\delta_\lambda^{(1,1)}\right)^2 = \frac{2}{3}\delta_\lambda^{(2,2)}. \tag{7.4.34}$$

In summary we find:

$$\beta(\lambda) = \lambda^2 \frac{\mathrm{d}}{\mathrm{d}\lambda} a_1(\lambda) \tag{7.4.35}$$

where:

$$Z_\lambda = 1 + \sum_{k=1}^{\infty} \frac{a_k(\lambda)}{\epsilon^k}. \tag{7.4.36}$$

A similar relation holds for γ_ϕ:

$$\gamma_\phi = -\frac{1}{2}\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} c_1(\lambda) \tag{7.4.37}$$

where:

$$Z_\phi = 1 + \sum_{k=1}^{\infty} \frac{c_k(\lambda)}{\epsilon^k}. \tag{7.4.38}$$

7.4.1 β and γ in $\phi^3_{d=6}$

Recall the values obtained in Caphter §3 for the $\overline{MS}$ scheme (here we convert to MS):

$$\delta_1 = -\frac{\lambda^2}{(4\pi)^3}\frac{1}{2\epsilon} + \mathcal{O}(\lambda^4) \tag{7.4.39}$$

$$= \lambda^2 \delta_1^{(1,1)} \frac{1}{\epsilon} + \mathcal{O}(\lambda^4) \tag{7.4.40}$$

$$\delta_\phi = -\frac{\lambda^2}{(4\pi)^3}\frac{1}{12\epsilon} + \mathcal{O}(\lambda^4) \tag{7.4.41}$$

$$= \lambda^2 \delta_\phi^{(1,1)} \frac{1}{\epsilon} + \mathcal{O}(\lambda^4) \tag{7.4.42}$$

so we find:

$$\delta_\lambda^{(1,1)} = \delta_1^{(1,1)} - \frac{3}{2}\delta_\phi^{(1,1)} = -\frac{1}{(4\pi)^3}\frac{3}{8} \tag{7.4.43}$$

and the beta function:

$$\beta(\lambda) = \lambda^2 \frac{\mathrm{d}}{\mathrm{d}\lambda} a_1(\lambda) \tag{7.4.44}$$

$$= \lambda^2 \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\lambda^2 \delta^{(1,1)} + \mathcal{O}(\lambda^4) \right) \tag{7.4.45}$$

$$= 2\lambda^3 \delta^{(1,1)} + \mathcal{O}(\lambda^5) \tag{7.4.46}$$

$$= -\frac{3}{4} \frac{\lambda^3}{(4\pi)^3} + \mathcal{O}(\lambda^5). \tag{7.4.47}$$

Exercise 15. Check that:

$$\gamma_\phi(\lambda) = \frac{1}{12} \frac{\lambda^2}{(4\pi)^3} + \mathcal{O}(\lambda^4). \tag{7.4.48}$$

Observation. Since $\beta(\lambda) < 0$ for small λ we have:

$$\mu \frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = \frac{\mathrm{d}\lambda}{\mathrm{d}\log(\mu)} < 0 \tag{7.4.49}$$

so, $\lambda(\mu)$ decreases as μ increases. See plot in figure 7.1.

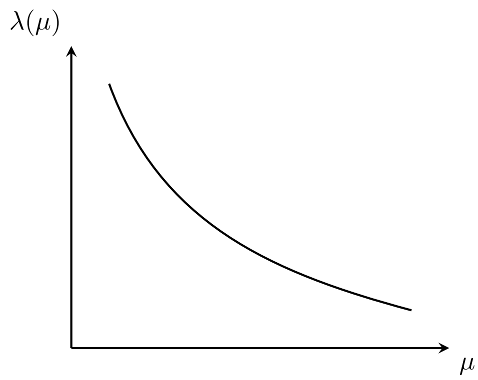


Figure 7.1

7.4.2 QED

We have:

$$\delta_e = \delta_1 - \delta_\psi - \frac{1}{2}\delta_A \tag{7.4.50}$$

$$= -\frac{1}{2}\delta_A \tag{7.4.51}$$

$$= \left(\frac{\alpha}{4\pi}\right) \frac{2}{3\epsilon} + \mathcal{O}(\alpha^2) \tag{7.4.52}$$

and since:

$$\alpha = \frac{e^2}{4\pi} \tag{7.4.53}$$

we get:

$$\delta_e = \frac{e^2}{16\pi^2}\frac{2}{3\epsilon} + \mathcal{O}(e^4). \tag{7.4.54}$$

The beta function, dependent on e, is:

$$\begin{aligned} \beta(e) &= \frac{e^3}{16\pi^2}\frac{4}{3} + \mathcal{O}(e^5) &(7.4.55)\\ &= e\left[\left(\frac{\alpha}{4\pi}\right)\frac{4}{3} + \mathcal{O}(\alpha^2)\right] &(7.4.56)\end{aligned}$$

but note that $\beta(e) \neq \beta(\alpha)$, and we have:

$$\begin{aligned} \beta(\alpha) &= \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\alpha(\mu) &(7.4.57)\\ &= \frac{1}{4\pi}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}e^2(\mu) &(7.4.58)\\ &= \frac{e}{2\pi}\beta(e) &(7.4.59)\\ &= 2\alpha\left[\left(\frac{\alpha}{4\pi}\right)\frac{4}{3} + \mathcal{O}(\alpha^2)\right]. &(7.4.60)\end{aligned}$$

Observation. We have $\beta(\alpha) > 0$ and for small α we have:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\log\mu} > 0 \tag{7.4.61}$$

and coupling increases with the scale. As you can see in figure 7.2.

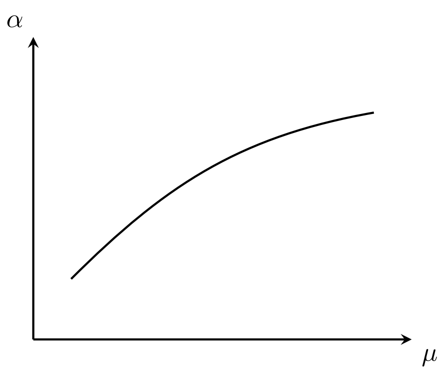


Figure 7.2

Exercise 16 S. ow the anomolous dimension are:

$$\gamma_\psi(e) = -\frac{1}{2}e\frac{\mathrm{d}c_1}{\mathrm{d}e} = \frac{e^2}{16\pi^2} + \mathcal{O}(e^4) \tag{7.4.62}$$

$$\gamma_A(e) = \frac{4}{3}\frac{e^2}{16\pi^2} + \mathcal{O}(e^4). \tag{7.4.63}$$

7.4.3 QCD

For QCD we can find:

$$\delta_{\psi_q} = \left(\frac{\alpha_s}{4\pi}\right)\frac{1}{\epsilon}(-C_F) + \mathcal{O}(\alpha_s^2) \tag{7.4.64}$$

$$\delta_A = \left(\frac{\alpha_s}{4\pi}\right)\frac{1}{\epsilon}(5C_A - 4T_R n_f)\frac{1}{3} + \mathcal{O}(\alpha_s^2) \tag{7.4.65}$$

$$\delta_1 = \left(\frac{\alpha_s}{4\pi}\right)\frac{1}{\epsilon}(-C_A - C_F) + \mathcal{O}(\alpha_s^2) \tag{7.4.66}$$

that implies:

$$\delta_{g_s} = \left(\frac{\alpha_s}{4\pi}\right)\frac{1}{\epsilon}(-11C_A + 4T_R n_f)\frac{1}{6} + \mathcal{O}(\alpha_s^2) \tag{7.4.67}$$

and:

$$\beta(g_s) = -g_s\left(\frac{\alpha_s}{4\pi}\right)\left(\frac{11C_A - 4T_R n_f}{3}\right) + \mathcal{O}(g_s^5) \tag{7.4.68}$$

$$\beta(\alpha_s) = -\alpha_s\left(\frac{\alpha_s}{4\pi}\right)\frac{2}{3}(11C_A - 4T_R b_f) + \mathcal{O}(\alpha_s^5) \tag{7.4.69}$$

$$= -\alpha_s\left[\left(\frac{\alpha_s}{2\pi}\right)b_0 + \mathcal{O}(\alpha_s^2)\right] \tag{7.4.70}$$

where:

$$b_0 = \frac{1}{3}(11C_A - 4T_R n_f). \tag{7.4.71}$$

Observation. For small α_s and for $n_f \leq 16$ we have:

$$\frac{\mathrm{d}\alpha_s}{\mathrm{d}\log\mu} < 0 \tag{7.4.72}$$

so the coupling decreases with the scale, and we have what is called *asymptotic freedom*, which indicates we may apply perturbation theory at high energies. See the figure 7.3.

7.5 Solution to the Callan-Symanzik equation

We start by reminding ourselves of the form of the 2-point correlation function in a scalar theory (massless):

$$\tilde{G}_2(p^2;\mu,\lambda(\mu)) = \frac{i}{p^2 + i\emptyset^+}g_2(-\mu^2/p^2,\lambda(\mu)) \tag{7.5.1}$$

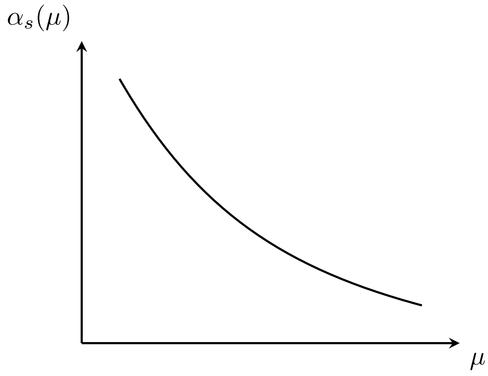


Figure 7.3

where:

$$g_s(-\mu^2/p^2, \lambda(\mu)) = 1 + \lambda^2 g_2^{(1)}(-\mu^2/p^2) + \mathcal{O}(\lambda^4). \tag{7.5.2}$$

Since g_2 depends only on the ratio $-\mu^2/p^2$, we can swap $\mu\frac{\partial}{\partial\mu}$ for $p^\mu\frac{\partial}{\partial p^\mu}$ in the CS equation:

$$\left(p^\mu \frac{\partial}{\partial p^\mu} - \beta(\lambda)\frac{\partial}{\partial\lambda} + 2(1-\gamma_\phi(\lambda))\right)\tilde{G}_2 = 0. \tag{7.5.3}$$

Proof. We can start noticing:

$$\mu\frac{\partial}{\partial\mu} = 2\mu^2\frac{\partial}{\partial\mu^2} \quad , \quad p^\mu\frac{\partial}{\partial p^\mu} = 2p^2\frac{\partial}{\partial p^2} \tag{7.5.4}$$

let $x = -\mu^2/p^2$ so that:

$$\mu^2\frac{\partial}{\partial\mu^2}g_2(x,\lambda) = \mu^2\frac{\partial x}{\partial\mu^2}\frac{\partial}{\partial x}g_2(x,\lambda) \tag{7.5.5}$$

$$= x\frac{\partial}{\partial x}g_2(x,\lambda) \tag{7.5.6}$$

$$p^2\frac{\partial}{\partial\mu^2}g_2(x,\lambda) = -x\frac{\partial}{\partial x}g_2(x,\lambda) \tag{7.5.7}$$

from which we show:

$$\mu\frac{\partial}{\partial\mu}\tilde{G}_2 = -\left(p^\mu\frac{\partial}{\partial p^\mu} + 2\right)\tilde{G}_2 \tag{7.5.8}$$

leading to the desired form of the CS equation.

Solution 1. We start with the free theroy, so $\beta = \gamma_\phi = 0$, and we have to solve:

$$\left(p^\mu \frac{\partial}{\partial p^\mu} + 2\right) \tilde{G}_2 = 0 \tag{7.5.9}$$

$$\implies \left(p^2 \frac{\partial}{\partial p^2} + 1\right) \tilde{G}_2 = 0 \tag{7.5.10}$$

$$\implies \tilde{G}_2 = \frac{k}{p^2}\ , \quad k = \text{constant}. \tag{7.5.11}$$

We can rewrite (7.5.10) using the mass dimension of the scalar field $\Delta = 1$:

$$p^2 \frac{\partial}{\partial p^2} \tilde{G}_2 = -\Delta \tilde{G}_2 \tag{7.5.12}$$

if we release the condition $\gamma = 0$ the CS equation is:

$$\frac{\partial}{\partial \log p^2} \tilde{G}_2 = -(1 - \gamma_\phi) \tilde{G}_2 \tag{7.5.13}$$

which demonstrates why γ_ϕ is called the anomalous dimension:

$$\Delta = 1 - \gamma_\phi \quad , \quad \tilde{G}_2 = \frac{k}{(p^2)^\Delta}. \tag{7.5.14}$$

Solution 2. We can talk about *Coleman's Hydrodynamic/Bacterialogical analogy* (see chapter §12, p. 418, in Peskin).

We consider a fluid flowing with velocity $v(x)$ along the axis of a narrow tube. As in figure 7.4.

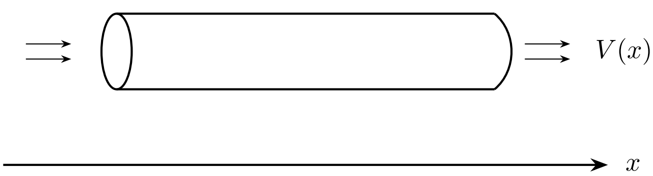


Figure 7.4

The liquid contains bacteria distributed according to a function $D(t, x)$ at time t and position x. The tube is illumated by light distributed according to a function $\rho(x)$. As you can see in figure 7.5.

The bacteria density will follow an evolution equation almost identical to the CS equation for correlation functions:

$$\left(\frac{\partial}{\partial t} + v(x) \frac{\partial}{\partial x} - \rho(x)\right) D(t, x) = 0. \tag{7.5.15}$$

The translation can be summarized as in table 7.1.

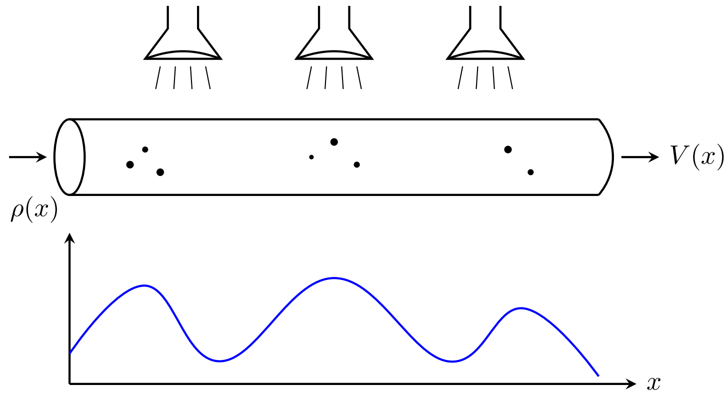


Figure 7.5

Tube model	CS equation
$\frac{\partial}{\partial t}$	$\frac{\partial}{\partial \log(\mu/\lvert p\rvert)}$
x	λ
$v(x)$	$-\beta(\lambda)$
$\rho(x)$	$2(\gamma(\lambda)-1)$
$D(t,x)$	$g_2(-p^2/\mu^2, \lambda(\mu))$

Table 7.1

We will find solutions according to the initial condition:

$$D(0,x) = D_{init}(x). \tag{7.5.16}$$

Let's solve some special cases first:

Static case: $v(x) = 0$, so we have:

$$\left(\frac{\partial}{\partial t} - \rho(x)\right) D(t,x) = 0 \tag{7.5.17}$$

which have solution:

$$D(t,x) = \exp\{\rho(x)t\} D_{init}(x). \tag{7.5.18}$$

No illumination: $\rho(x) = 0$, so we have:

$$\left(\frac{\partial}{\partial t} + v(x)\frac{\partial}{\partial x}\right) D(t,x) = 0 \tag{7.5.19}$$

in the case $v(x) = v$ is a constant, we find a simple travelling wave solution:

$$D(t,x) = D(x - vt) \quad , \qquad z = x - vt \tag{7.5.20}$$

$$\frac{\partial}{\partial t} D(z) = -vD'(z) \quad , \qquad \frac{\partial}{\partial x} D(x) = D'(z) \tag{7.5.21}$$

so:

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) D(z) = 0. \tag{7.5.22}$$

General case. For the general case we may write the solution using a fluid element which moves according ot $\hat{x}(t)$. At a time $t = 0$ we find the element at x_0:

$$\hat{x}(0) = x_0. \tag{7.5.23}$$

Since the β function is $-v$ in the analogy we define $\hat{x}(t)$ to be the position for $t < 0$. The fluid element is described by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t) = -v(\hat{x}(t)) \ , \quad \hat{x}(0) = x_0 \tag{7.5.24}$$

which has the solution:

$$\int_{x_0}^{\hat{x}(t)} \frac{\mathrm{d}\hat{x}}{v(\hat{x})} = -\int_0^t \mathrm{d}t' = -t. \tag{7.5.25}$$

The dependence on the initial condition can be denoted $\hat{x} \equiv \hat{x}(t; x_0)$. Let's change the initial condition to be somewhere arbitrary $x_0 \to x$ and find $\hat{x}(t; x)$ that satisfies both:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t; x) = -v(\hat{x}(t, x)) \tag{7.5.26}$$

$$\left(\frac{\partial}{\partial t} + v(x)\frac{\partial}{\partial x}\right) \hat{x}(t; x) = 0. \tag{7.5.27}$$

To clarify we introduce a primitive:

$$P = \frac{\mathrm{d}\hat{x}}{v(\hat{x})} \tag{7.5.28}$$

with inverse U such that:

$$U(P(y)) = y \tag{7.5.29}$$

and:

$$-t = \int_x^{\hat{x}(t;x)} \frac{\mathrm{d}\hat{x}}{v(\hat{x})} \tag{7.5.30}$$

$$= \int_x^{\hat{x}(t;x)} \mathrm{d}P \tag{7.5.31}$$

$$= P(\hat{x}(t; x)) - P(x) \tag{7.5.32}$$

so we get:

$$\hat{x}(t; x) = U(P(x) - t) \tag{7.5.33}$$

which solves (7.5.26). Now check (7.5.27):

$$\frac{\partial \hat{x}}{\partial t} = \frac{\partial}{\partial t}(P(x)-t)U'(P(x)-t) \tag{7.5.34}$$
$$= -U'(P(x)-t) \tag{7.5.35}$$
$$\frac{\partial \hat{x}}{\partial x} = \frac{\partial}{\partial x}(P(x)-t)U'(P(x)-t) \tag{7.5.36}$$
$$= \frac{\mathrm{d}P(x)}{\mathrm{d}x}U'(P(x)-t) \tag{7.5.37}$$
$$= \frac{1}{v(x)}U'(P(x)-t) \tag{7.5.38}$$
$$= \frac{1}{v(x)}\left(-\frac{\partial \hat{x}}{\partial t}\right) \tag{7.5.39}$$

which confirms equations (7.5.27) is satisfied. We can build the solution for $D(x,t)$ using $\hat{x}(t;x)$:

$$D(x,t) = D_{init}(\hat{x}(t;x)) \tag{7.5.40}$$

where:

$$D_{init}(x) = D_{init}(\hat{x}(0;x)) \tag{7.5.41}$$
$$= D(0,x). \tag{7.5.42}$$

Verification. We start from:

$$0 = \left(\frac{\partial}{\partial t} + v(x)\frac{\partial}{\partial x}\right)D(t,x) \tag{7.5.43}$$
$$= \frac{\partial}{\partial t}D_{init}(\hat{x}(t;x)) + v(x)\frac{\partial}{\partial x}D_{init}(\hat{x}(t;x)) \tag{7.5.44}$$
$$= \frac{\partial \hat{x}}{\partial t}D'_{init} + v(x)\frac{\partial \hat{x}}{\partial x}D'_{init} \tag{7.5.45}$$
$$= D'_{init}\left(\frac{\partial \hat{x}}{\partial t} + v(x)\frac{\partial \hat{x}}{\partial x}\right) \tag{7.5.46}$$
$$\overset{(7.5.27)}{=} 0. \tag{7.5.47}$$

General case: with $\rho(x) \neq 0$ and $v(x) \neq 0$. The general case is a combination of the solutions found already:

$$D(t,x) = D_{init}(\hat{x}(t;x))\exp\left\{\int_0^t \mathrm{d}t'\,\rho(\hat{x}(t;x))\right\} \tag{7.5.48}$$
$$\equiv f_1(t,x)f_2(t,x). \tag{7.5.49}$$

The solution is straightforward to verify, calling:

$$O(t,x) = \frac{\partial}{\partial t} + v(x)\frac{\partial}{\partial x} \tag{7.5.50}$$

so:

$$0 = \Big(O(t,x) - \rho(x)\Big) D(t,x) \tag{7.5.51}$$

$$= \underbrace{O(f_1)}_{=0} f_2 + f_1 O(f_2) - \rho f_1 f_2 \tag{7.5.52}$$

$$= f_1 \left[\rho(\hat{x}) f_2 + \left(\int_0^t \mathrm{d}t'\, v(x) \frac{\partial}{\partial x} \rho(\hat{x})\right) f_2\right] - f_1 f_2 \rho(x) \tag{7.5.53}$$

$$= f_1 f_2 \left(\rho(\hat{x}) - \rho(x) + \int_0^t \mathrm{d}t'\, v(x) \frac{\partial}{\partial x} \rho(\hat{x})\right). \tag{7.5.54}$$

Now, recall that:

$$O(t,x)\hat{x}(t;x) = 0 \tag{7.5.55}$$

$$\implies \quad v(x) \frac{\partial}{\partial x} \rho(\hat{x}) = -\frac{\partial}{\partial t} \rho(\hat{x}) \tag{7.5.56}$$

$$\implies \quad \int_0^t \mathrm{d}t'\, v(x) \frac{\partial}{\partial x} \rho(\hat{x}(t';x)) = -\int_0^t \mathrm{d}t'\, \frac{\partial}{\partial t'} \rho(\hat{x}(t';x)) \tag{7.5.57}$$

$$= -\rho(\hat{x}(t;x)) + \rho(\hat{x}(0;x)) \tag{7.5.58}$$

$$= -\rho(\hat{x}) + \rho(x) \tag{7.5.59}$$

which completes the validation of our solution.

NB. An alternative form of the solution is:

$$D(t,x) = D_{init}(\hat{x}(t;x)) \exp\left\{-\int_x^{\hat{x}(t;x)} \mathrm{d}x'\, \frac{\rho(x')}{v(x')}\right\}. \tag{7.5.60}$$

Translating the solution back to the 2-point correlation function leads to:

$$\tilde{G}_2(p,\mu,\lambda(\mu)) = G_{init}(\mu, \hat{\lambda}(p;\lambda)) \exp\left\{-\int_{|p'|=\mu}^{|p'|=|p|} \mathrm{d}\log\left(\frac{|p'|}{\mu}\right) 2(1 - \gamma_\phi(\hat{\lambda}(p;\lambda)))\right\} \tag{7.5.61}$$

where:

$$\frac{\mathrm{d}}{\mathrm{d}\log\left(\frac{|p|}{\mu}\right)} \hat{\lambda}(p;\lambda) = \beta(\hat{\lambda}). \tag{7.5.62}$$

Observation. Since:

$$\exp\left\{-2\int \mathrm{d}\log\left(\frac{|p'|}{\mu}\right)\right\} = \frac{\mu^2}{p^2} \tag{7.5.63}$$

we can write:

$$\tilde{G}_2(p,\mu,\lambda(\mu)) = \frac{-i}{p^2} g_2'(\mu, \hat{\lambda}(p;\lambda)) \exp\left\{2\int \mathrm{d}\log\left(\frac{|p'|}{\mu}\right) \gamma_\phi(\hat{\lambda})\right\} \tag{7.5.64}$$

$$= \frac{-i}{p^2} g_2(-\mu^2/p^2, \hat{\lambda}(p;\lambda)) \tag{7.5.65}$$

where $\hat{\lambda}(\mu;\lambda) = \lambda$.

7.6 Running couplings

Let's explore the solutions α and α_s using the β function evaluated up to 1-loop. Ww saw:

$$\beta(\alpha_s) = -\left(\frac{\alpha_s^2}{2\pi}\right) b_0 + \mathcal{O}(\alpha_s^3) \tag{7.6.1}$$

where:

$$b_0 = \frac{11C_A - 4T_R n_f}{3}. \tag{7.6.2}$$

So we can see:

$$\implies \quad \frac{\mathrm{d}\alpha_s}{\alpha_s^2} = -\frac{b_0}{2\pi}\frac{\mathrm{d}\mu}{\mu} \tag{7.6.3}$$

$$\implies \quad \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{\mathrm{d}\alpha_s}{\alpha_s^2} = -\frac{b_0}{2\pi}\int_{\mu_0}^{\mu}\frac{\mathrm{d}\mu'}{\mu'} \tag{7.6.4}$$

$$\implies \quad \left[-\frac{1}{\alpha_s}\right]_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} = -\frac{b_0}{2\pi}\log\left(\frac{\mu}{\mu_0}\right) \tag{7.6.5}$$

$$\implies \quad -\frac{1}{\alpha(\mu)} + \frac{1}{\alpha(\mu_0)} = -\frac{b_0}{2\pi}\log\left(\frac{\mu}{\mu_0}\right) \tag{7.6.6}$$

$$\implies \quad \alpha(\mu) = \frac{\alpha(\mu_0)}{1 + \frac{b_0}{2\pi}\log\left(\frac{\mu}{\mu_0}\right)\alpha(\mu_0)} \tag{7.6.7}$$

so we can write the coupling as:

$$\alpha(\mu) = \frac{\alpha(\mu_0)}{1 + \frac{b_0}{4\pi}\log\left(\frac{\mu^2}{\mu_0^2}\right)\alpha(\mu_0)}. \tag{7.6.8}$$

To study the QED evolution use $T_R = 1$, $C_A = 0$ and $n_f = 1$, so $b_0^{QED} = -4/3$. The $\alpha(\mu)$ plot is the figure 7.6.

NB. α in the Standard Model recives corrections from quarks a W/Z boson, when above threshold:

$$\alpha^{SM}(M_Z) \approx \frac{1}{128}. \tag{7.6.9}$$

If we want to see the QCD evolution, which have $N_c = 3$, $n_f = 5$ and $T_R = 1/2$, so $b_0 = 23/3$, we can see the figure 7.7.

Observation. The running coupling has a pole at:

$$1 + \left(\frac{b_0}{4\pi}\right)\alpha(\mu_0)\log\left(\frac{\mu_L^2}{\mu_0^2}\right) = 0 \tag{7.6.10}$$

where μ_L^2 is the location of the *Landau pole*. In QED we find:

$$\mu_L^{QED} \approx m_e \exp\{650\} \tag{7.6.11}$$

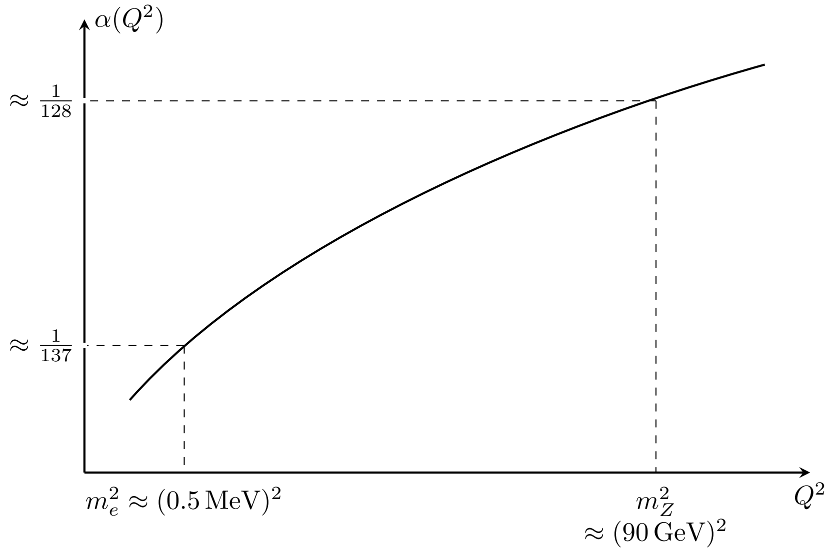


Figure 7.6

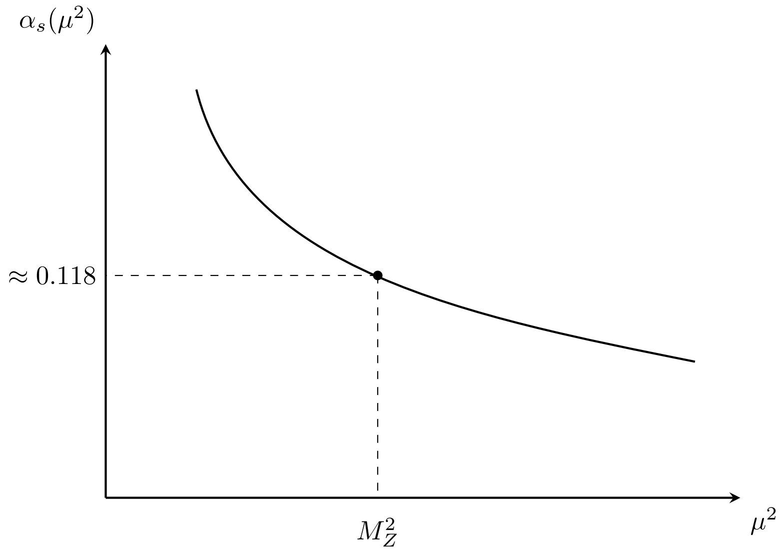


Figure 7.7

ans so way above any physical scale. For QCD the Landau pole is inside the confined region where $\alpha_s > 1$:

$$\mu_L^{QCD} = M_Z \exp\left\{-\frac{2\pi}{b_0 \alpha_s(M_Z)}\right\} \tag{7.6.12}$$

$$\approx 250 \text{ MeV} \quad \text{for } n_f = 3 \text{ active flavour.} \tag{7.6.13}$$

7.6.1 Higher order running couplings

Analytic solution to the running coupling at higher orders are not available in general. Numerical solutions may be obtained. We may find analytic expressions when keep only the dominant logarithmic contributions. We find:

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\alpha = -\alpha\left(\frac{\alpha}{2\pi}b_0 + \left(\frac{\alpha}{2\pi}\right)^2 b_1 + \mathcal{O}(\alpha^3)\right) \tag{7.6.14}$$

$$= -\alpha^2\beta_0\left(1 + \alpha\frac{\beta_1}{\beta_0} + \mathcal{O}(\alpha^2)\right) \tag{7.6.15}$$

where:

$$\beta_k = \frac{b_k}{(2\pi)^{k+1}}. \tag{7.6.16}$$

So we find:

$$\implies \quad -\frac{\mathrm{d}\mu}{\mu} = \frac{\mathrm{d}\alpha}{\alpha^2\beta_0}\frac{1}{1+\alpha\frac{\beta_1}{\beta_0}} \tag{7.6.17}$$

$$\implies \quad -\log\left(\frac{\mu}{\mu_0}\right) = c - \frac{1}{\beta_0\alpha} - \frac{\beta_1}{\beta_0^2}\left[\log(\alpha) - \log\left(1+\alpha\frac{\beta_1}{\beta_0}\right)\right] \tag{7.6.18}$$

where c is a constant of integration. We can finf the solution interactively. First set $\beta_1 = 0$ and choose c such that:

$$-\log\left(\frac{\mu}{\Lambda}\right) = -\frac{1}{\alpha\beta_0} \equiv -L \quad , \quad c = \log\left(\frac{\mu_0}{\Lambda}\right). \tag{7.6.19}$$

In other words:

$$\alpha = \frac{1}{\beta_0 L}. \tag{7.6.20}$$

Now we take the leading log limit of the two loop equation, correcting the integration constant:

$$c = \log\left(\frac{\mu_0}{\Lambda}\right) - 2\frac{\beta_1}{\beta_0^2}\log(\beta_0) \tag{7.6.21}$$

which leads to (substituting $\alpha = 1/(\beta_0 L)$ into the argument of log then expand):

$$-L = -\frac{1}{\alpha\beta_0} + \frac{\beta_1}{\beta_0^2}\log L \tag{7.6.22}$$

so we find:

$$\alpha = \frac{1}{\beta_0 L}\left(1 - \frac{\beta_1}{\beta_0}\frac{\log L}{L}\right) + \mathcal{O}\left(\frac{1}{L^3}\right). \tag{7.6.23}$$

In QCD one finds that:

$$b_1 = \frac{34}{3}C_A^2 - \frac{20}{3}C_A T_R n_f - 4C_F T_R n_f \tag{7.6.24}$$

and the effect of $2L$ to IL running can be $1-2\%$ (for modern collider energy scales).

7.6.2 All order behaviour of β

We saw for $\alpha \ll 1$ the plot in figure 7.8.

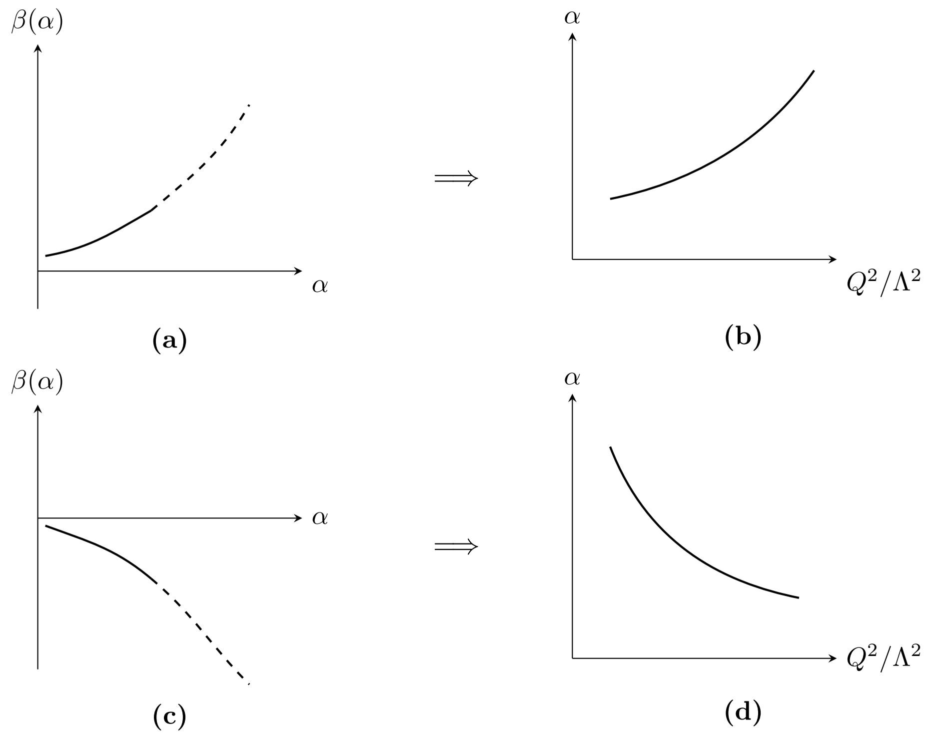


Figure 7.8

But, we can ask yourself what happens for larger values of α. It is possible that $\beta'(\alpha) = 0$ at some point and we find $\beta(\alpha_*) = 0$, where $\alpha^* > 0$, and we can see the plot in figure 7.9a, the IR fixed point, and in figure 7.9b, the UV fixed point.

If we assume such a point does exsist we can infer the properties of the theory close to the fixed point. Take:

$$\beta(\alpha) = -B(\alpha - \alpha_*) \tag{7.6.25}$$

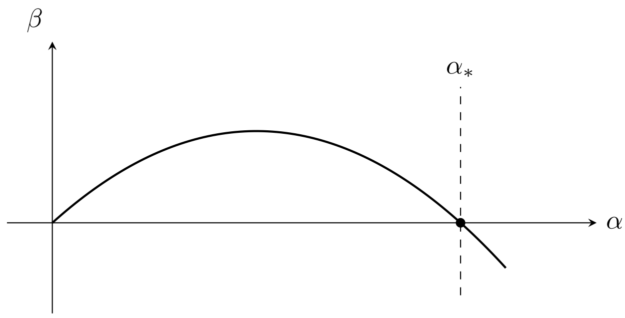
close to α^* where B is a constant. We can see:

$$\int_{\alpha(\mu)}^{\alpha(|p|)} \frac{\mathrm{d}\alpha}{\alpha - \alpha_*} = -B \log\left(\frac{|p|}{\mu}\right) \tag{7.6.26}$$

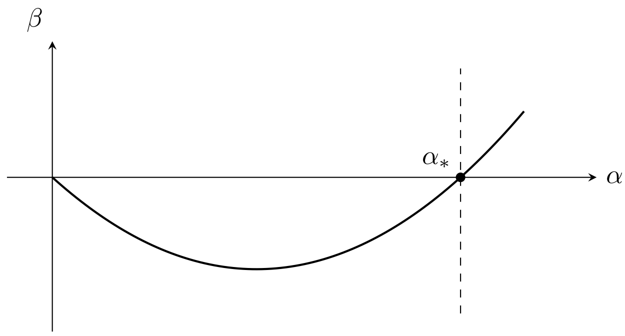
for same scales $|p|$ and μ. We get:

$$\Longrightarrow \quad \log\left(\frac{\alpha(|p|) - \alpha_*}{\alpha(\mu) - \alpha_*}\right) = \log\left(\left(\frac{|p|}{\mu}\right)^{-B}\right) \tag{7.6.27}$$

$$\Longrightarrow \quad \alpha(|p|) = \alpha_* + (\alpha(\mu) - \alpha_*)\left(\frac{|p|}{\mu}\right)^{-B} \tag{7.6.28}$$



(a) IR fixed point.



(b) UV fixed point. It's useful for studing phase transitions in e.g. Ising model.

Figure 7.9

so we find an exact solution to the evolution of the coupling. We can also look at the solution to the 2-point function from the CS equation. Take:

$$\gamma_* = \gamma(\alpha(|p|; \alpha(\mu))) \tag{7.6.29}$$

so:

$$\tilde{G}_2(p^2, \mu^2, \alpha_*) = \frac{-i}{p^2} g_2'(\alpha_*) \exp\left\{ 2 \int_{\mu}^{|p|} \frac{\mathrm{d}|p'|}{|p'|} \gamma_* \right\} \tag{7.6.30}$$

$$= \frac{-i}{p^2} \left(\frac{\mu^2}{p^2} \right)^{-\gamma_*} g_2'(\alpha_*). \tag{7.6.31}$$

7.7 Evolution of mass parameters

Let's consider a massive scalar theory and look at the CS equation for an on-shell ($p_i^2 = m^2$) S-matrix element. In the on-shell case the factor of $n\gamma_\phi(\lambda)$ does not contribute (LSZ):

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} + \gamma_m(\lambda) m \frac{\partial}{\partial m} \right) S = 0 \tag{7.7.1}$$

where:

$$S \equiv S(p_i \cdot p_j, m; \mu, \lambda(\mu)) \tag{7.7.2}$$

with $p_i \cdot p_j$ the momentum invariants. We can normalize S to form a dimensionless function $\hat{S}$ using the scale μ:

$$S = \mu^{-d} \hat{S} \left(\frac{p_i \cdot p_j}{\mu^2}, \frac{m^2}{\mu^2}, \lambda(\mu) \right). \tag{7.7.3}$$

We can perform a simple dimensional analysis of S, taking a single invariant Q^2 for simplicity:

$$\hat{S} \equiv \hat{S} \left(\frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}, \lambda(\mu) \right). \tag{7.7.4}$$

So we get the equation:

$$\left(\mu \frac{\partial}{\partial \mu} + Q \frac{\partial}{\partial Q} + m \frac{\partial}{\partial m} + d \right) S = \tag{7.7.5}$$

$$= 2 \left(\mu^2 \frac{\partial}{\partial \mu^2} + Q^2 \frac{\partial}{\partial Q^2} + m^2 \frac{\partial}{\partial m^2} + \frac{d}{2} \right) (\mu^2)^{-d/2} \hat{S} \tag{7.7.6}$$

$$= 2 (\mu^2)^{-d/2} \left(\mu^2 \frac{\partial}{\partial \mu^2} + Q^2 \frac{\partial}{\partial Q^2} + m^2 \frac{\partial}{\partial m^2} \right) \hat{S} \tag{7.7.7}$$

$$= 0. \tag{7.7.8}$$

Combining (7.7.5) minus (7.7.1) gives:

$$\left(Q \frac{\partial}{\partial Q} - \beta(\lambda) \frac{\partial}{\partial \lambda} + (1 - \gamma_m(\lambda)) m \frac{\partial}{\partial m} + d \right) S = 0. \tag{7.7.9}$$

This equation may be solved using the same methos as used in section §7.5 and leads to a solution in which both coupling and mass run with energy:

$$\hat{\lambda}(Q; \lambda) \quad \text{satisfying} \quad Q \frac{\mathrm{d}\hat{\lambda}}{\mathrm{d}Q} = \beta(\hat{\lambda}) \tag{7.7.10}$$

$$\hat{m}(Q; \lambda) \quad \text{satisfying} \quad Q \frac{\mathrm{d}\hat{m}}{\mathrm{d}Q} = -(1 - \gamma_m(\hat{\lambda})) \hat{m} \tag{7.7.11}$$

where $\hat{\lambda}$ runs from the point λ where we habe measured the coupling and fixed the renormalization scheme. The mass runs from the value of the mass at the same renormalization scale, μ for example:

$$\hat{\lambda}(\mu; \lambda) = \lambda(\mu) \tag{7.7.12}$$

$$\hat{m}(\mu; \lambda) = m(\mu) \tag{7.7.13}$$

then:

$$\hat{m}(Q; \lambda) = m(\mu) \exp \left\{ - \int_\mu^Q \frac{\mathrm{d}Q'}{Q'} (1 - \gamma_m(\hat{\lambda})) \right\}. \tag{7.7.14}$$

Chapter 8

Unitarity and gauge invariance

Unitarity of the S-matrix leads to powerful constraints on the discontinuities of loop amplitudes obtained from lower order amplitudes. We will compute discontinuities/generalised discontinuities via Cutkosky rules, and that will leads to *generalized unitarity* constraints, that can be used to determine the coefficients of basis loop integrals for any amplitude.

The unitarity of the S-matrix for physical states relies on the cancellation of unphysical degrees of freedom (ghosts and longitudinal polarisations). This can be demostrated using BRST symmetry.

8.1 $SS^\dagger = 1$ and implications

In the Heisemberg picture, where operators are time dependent and states are time independent, we may take (postulate) two orthonormal bases for the Hilbert (/Fock) space:

$$|\alpha, \text{in}\rangle \qquad \text{observations at } t \to -\infty \tag{8.1.1}$$

$$|\alpha, \text{out}\rangle \qquad \text{observations at } t \to +\infty. \tag{8.1.2}$$

The scattering matrix S has elements:

$$S_{\beta\alpha} = \langle \beta, \text{out} | \alpha, \text{in} \rangle \tag{8.1.3}$$

and, since *in* states are orthonormal, we have:

$$\langle \alpha, \text{in} | \beta, \text{in} \rangle = \delta_{\alpha\beta}. \tag{8.1.4}$$

We may indsert a complete set of *out* states in the relation above, to find:

$$\sum_k \langle \alpha, \text{in} | k, \text{out} \rangle \langle k, \text{out} | \beta, \text{in} \rangle = \sum_k (S_{k\alpha})^\dagger S_{k\beta} \tag{8.1.5}$$

$$= \delta_{\alpha\beta} \tag{8.1.6}$$

or simply:

$$S^\dagger S = SS^\dagger = 1. \tag{8.1.7}$$

The *optical theorem* connects this relation to the fact that the immaginary part of the forward scattering amplitude is related to the cross-section. There is a more general connection the *discontinuities* of loop amplitudes. Let:

$$S = \mathbb{1} + iT \tag{8.1.8}$$

where T is the transition matrix. Inserting this into $SS^\dagger = 1$ leads to:

$$-i(T - T^\dagger) = TT^\dagger = T^\dagger T. \tag{8.1.9}$$

The scattering amplitude $\mathcal{A}$ from a state I to a state F is defined by:

$$\langle F| T |I\rangle = i(2\pi)^4\delta^{(4)}(p_I - p_F)\mathcal{A}(I \to F) \tag{8.1.10}$$

where p_I is the total incoming momentum and p_F is the total outgoing. Combining (8.1.9) and (8.1.10) gives:

$$-i\langle F| T |I\rangle + i\langle F| T^\dagger |I\rangle = \langle F| T^\dagger T |I\rangle \tag{8.1.11}$$

$$= \sum\!\!\!\!\!\!\int_k \langle F| T^\dagger |k\rangle \langle k| T |I\rangle \tag{8.1.12}$$

where we use:

$$\sum\!\!\!\!\!\!\int_k = \sum_{n=2}^{\infty} \int \prod_{j=1}^{n} \frac{\mathrm{d}^3\vec{k}_j}{(2\pi)^3 2E_j} \quad , \quad E_j^2 = |\vec{k}_j|^2 + m_j^2 \tag{8.1.13}$$

and we get:

$$= (2\pi)^4\delta^{(4)}(p_I - p_F)\mathcal{A}(I \to F) - (2\pi)^4\delta^{(4)}(p_I - p_F)\mathcal{A}(F \to I) \tag{8.1.14}$$

$$= \sum\!\!\!\!\!\!\int_k \Big(i(2\pi)^4\delta^{(4)}(p_k - p_F)\mathcal{A}(F \to k)\Big)^\dagger (2\pi)^4\delta^{(4)}(p_k - p_I)\mathcal{A}(I \to k) \tag{8.1.15}$$

$$= \sum\!\!\!\!\!\!\int_k (2\pi)^8\delta^{(4)}(p_k - p_F)\delta^{(4)}(p_k - p_I)\mathcal{A}(F \to k)^\dagger\mathcal{A}(I \to k) \tag{8.1.16}$$

$$= \sum\!\!\!\!\!\!\int_k (2\pi)^8\delta^{(4)}(p_I - p_F)\delta^{(4)}(p_k - p_I)\mathcal{A}(F \to k)^\dagger\mathcal{A}(I \to k) \tag{8.1.17}$$

hence:

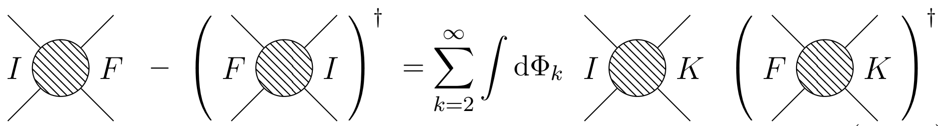
$$\mathcal{A}(I \to F) - \mathcal{A}(F \to I)^\dagger = \sum\!\!\!\!\!\!\int_k (2\pi)^4 \delta^{(4)}(p_k - p_I) \mathcal{A}(F \to k)^\dagger \mathcal{A}(I \to k). \tag{8.1.18}$$

If we set $F = I$ we recover the optical theorem:

$$2i\,\mathrm{Im}\{\mathcal{A}(I \to I)\} = \sum\!\!\!\!\!\!\int_k (2\pi)^4 \delta^{(4)}(p_k - p_I) |\mathcal{A}(I \to k)|^2 \tag{8.1.19}$$

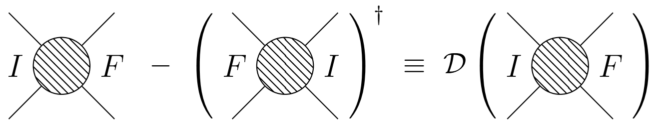
$$= \sum_{k=2}^{\infty} \int \mathrm{d}\Phi_k(p_I; p_1, \ldots, p_k)\, |\mathcal{A}(I \to k)|^2 \tag{8.1.20}$$

where we wrote the k-particle phase-space integral. We may represent relation (8.1.18) graphically by:



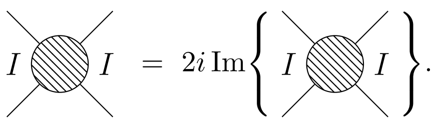
(8.1.21)

where we define:



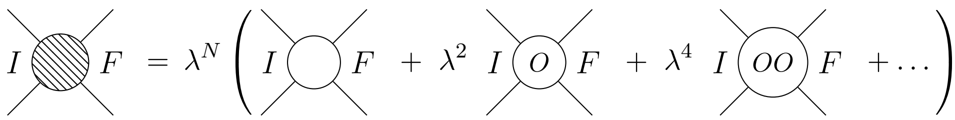
(8.1.22)

e.g.:

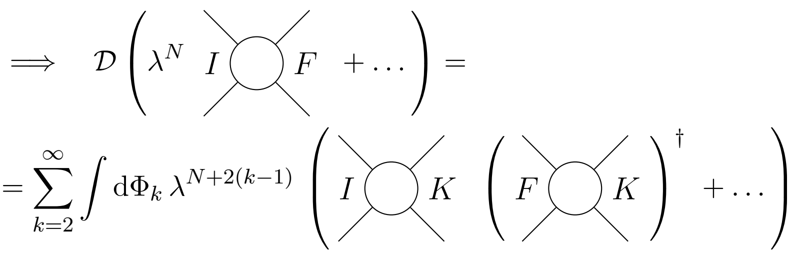


(8.1.23)

The amplitudes can be expanded perturbatively. If we assume a theory with a 3-point coupling λ than the leading order will be λ^n where $N = \#I + \#F - 2$. Therefore:

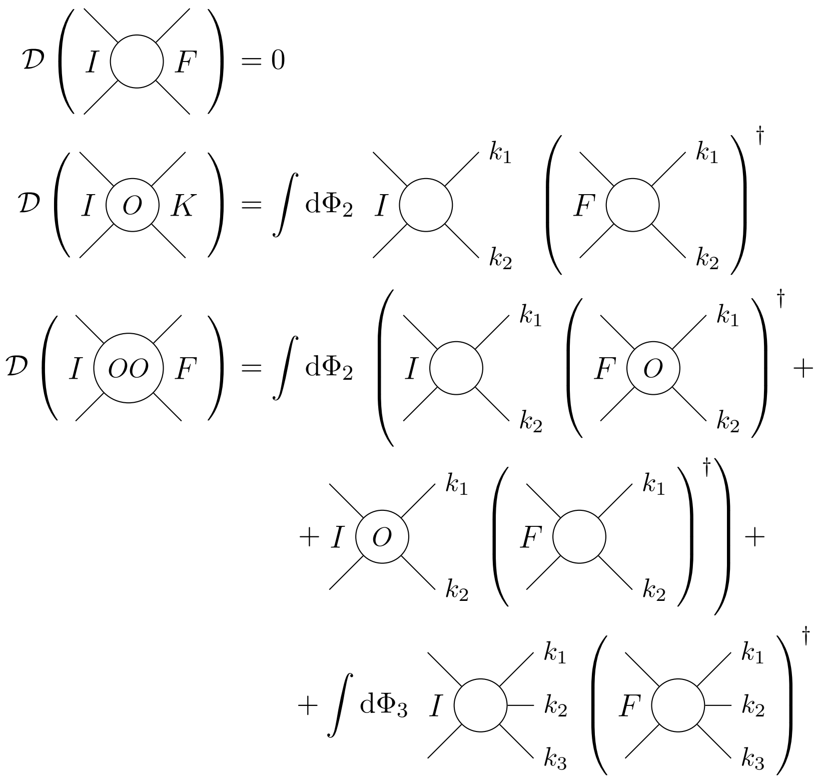


that implies:



(8.1.24)

Comparing order by order in λ gives:



so we find relations between amplitudes at different loop orders. The operation $\mathcal{D}$ is not $2i\,\mathrm{Im}$ in general since the amplitude contains *branch cuts* which lead to *discontinuities*. Let's look at a particular example to understand better.

8.1.1 Discontinuities of Feynman diagrams

The discontinuity of an amplitude $\mathcal{A}$ across the p^2-channel is defined by:

$$\lim_{\epsilon\to 0}\Big(\mathcal{A}(\dots,p^2+i\epsilon,\dots)-\mathcal{A}(\dots,p^2-i\epsilon,\dots)\Big)\equiv \mathrm{Disc}_{p^2}(\mathcal{A}). \tag{8.1.28}$$

It is useful to look at the branch cut in the logarithm to see how this is connected to the operation $\mathcal{D}$:

$$\mathrm{Disc}_x(\log{(x)}) = \lim_{\epsilon\to 0}\Big(\log{(x+i\epsilon)}-\log{(x-i\epsilon)}\Big) \tag{8.1.29}$$

$$= \begin{cases} 0 & , \quad x>0 \\ 2\pi i & , \quad x<0. \end{cases} \tag{8.1.30}$$

For $x<0$ we have $\log{(x)}=\log{(|x|)}+i\pi$ so we may write:

$$2i\,\mathrm{Im}\{\log{(x)}\} = \mathrm{Disc}_x(\log{(x)}) \qquad , \quad x<0. \tag{8.1.31}$$

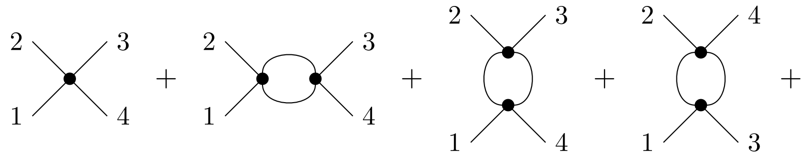
This helps to justift that for general states I, F the operation $\mathcal{D}$ *is the discontinuity of the amplitude* in the $I \to F$ channel.

For concreteness let's choose a $2 \to 2$ scattering amplitude $\mathcal{A}(p_1 p_2 \to p_3 p_4)$ and consider the discontinuity in the $s_{12} = (p_1 + p_2)^2$ channel:

$$\text{Disc}_{s_{12}}\Big(\mathcal{A}(p_1 p_2 \to p_3 p_4)\Big) = \mathcal{A}(p_1 p_2 \to p_3 p_4) - \mathcal{A}(p_3 p_4 \to p_1 p_2)^\dagger \quad (8.1.32)$$

$$= \sum_{k=1}^{\infty} \int \mathrm{d}\Phi_k \; \mathcal{A}(p_1 p_2 \to \{k\})\big(\mathcal{A}(p_3 p_4 \to \{k\})\big)^\dagger . \quad (8.1.33)$$

In ϕ^4 theory we can even explicitly write the amplitude up to 1-loops as:

$\mathcal{A}_4 =$ 

$$+ \text{counter-terms} + \mathcal{O}(\lambda^4) \quad (8.1.34)$$

so:

$$i\mathcal{A}_4^{[d]} = \lambda + \frac{\lambda^2}{2}\left[\hat{I}_2^{[d]}\left(\frac{s_{12}}{m^2}, \frac{\mu_R^2}{m^2}\right) + \hat{I}_2^{[d]}\left(\frac{s_{14}}{m^2}, \frac{\mu_R^2}{m^2}\right) + \hat{I}_2^{[d]}\left(\frac{s_{13}}{m^2}, \frac{\mu_R^2}{m^2}\right)\right] + \mathcal{O}(\lambda^4) \quad (8.1.35)$$

where:

$$\hat{I}_2^{[d]} = -i\mu_R^{2\epsilon} I_2 \Big|_{\text{finite}} \quad (8.1.36)$$

in a minimal scheme. We already know:

$$\hat{I}_2^{[d]}\left(\frac{s_{12}}{m^2}, \frac{\mu_R^2}{m^2}\right) = \frac{1}{(4\pi)^2} C_\Gamma \left(\frac{\mu_R^2}{m^2}\right)^\epsilon \left(\frac{1}{\epsilon} + 2 - \beta \log\left(\frac{-\beta_+}{\beta_-}\right)\right) - \frac{1}{(4\pi)^2}\frac{C_\Gamma}{\epsilon} \quad (8.1.37)$$

$$= \frac{1}{(4\pi)^2}\left(2 - \beta \log\left(\frac{-\beta_+}{\beta_-}\right) + \log\left(\frac{\mu_R^2}{m^2}\right)\right) \quad (8.1.38)$$

with:

$$\beta = \sqrt{1 - \frac{4m^2}{s_{12}}}. \quad (8.1.39)$$

The relevant discontinuity in $\mathcal{A}_4$ in the s_{12}-channels comes from the s_{12} bubble integral so we can explicitly study the discontinuity via poles in the integrand. We have:

$$I_2^{[d]}(p^2, m^2) = \int_k \frac{1}{((k - p/2)^2 - m^2 + i0^+)((k + p/2)^2 - m^2 + i0^+)} \quad (8.1.40)$$

which contains four possible poles. See the figure 8.1. Explicitly, choosing the frame $p = (p^0, \vec{\emptyset})$:

$$(k \pm p/2)^2 - m^2 + i\emptyset^+ = (k^0)^2 - |\vec{k}|^2 \pm k^0 p^0 + \frac{(p^0)^2}{4} - m^2 + i\emptyset^+ \quad (8.1.41)$$
$$= (k^0)^2 \pm k^0 p^0 + B \quad (8.1.42)$$
$$= (k_0 - k_\pm^+)(k_0 - k_\pm^-) \quad (8.1.43)$$

where we use:

$$B = -|\vec{k}|^2 + \frac{(p^0)^2}{4} - m^2 + i\emptyset \quad (8.1.44)$$
$$k_\delta^\sigma = \frac{1}{2}\left(\delta p^0 + \sigma\sqrt{p_0^2 - 4B}\right). \quad (8.1.45)$$

The poles in the complex k^0 plane are located, as already said, in the figure 8.1.

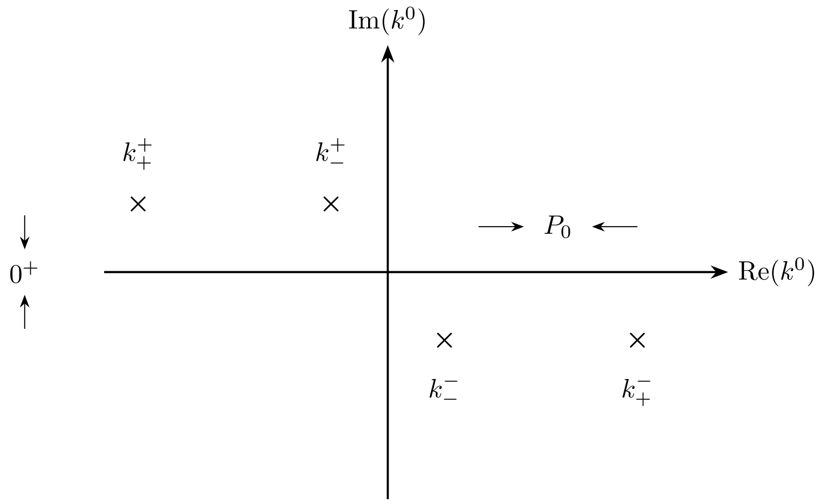


Figure 8.1

We now perform the k^0 integration ($4d$ only):

$$\int_k \longrightarrow \int \frac{d^3\vec{k}}{(2\pi)^3} \oint_\gamma \frac{dk^0}{2\pi} \quad (8.1.46)$$

using the residue theorem. You can see che curve and the regions in figure 8.2. At $k^0 = k_-^-$ we have:

$$\text{Res}_{k^0=k_-^-}(I_2^{[4]}) = \frac{1}{2\pi}\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{(k_-^- - k_+^-)(k_-^- - k_+^+)(k_-^- - k_-^+)} \quad (8.1.47)$$
$$= \frac{1}{2\pi}\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{(-2E_k + i\emptyset^+)p_0(-2E_k + p_0 + i\emptyset^+)} \quad (8.1.48)$$

where $E_k = |\vec{k}|^2 + m^2$. Since:

$$\frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} = \int \frac{\mathrm{d}|\vec{k}|\ |\vec{k}|^2\ \mathrm{d}^2\Omega}{(2\pi)^3} \tag{8.1.49}$$

$$= \int_m^\infty \frac{\mathrm{d}E_k\ |\vec{k}|\ E_k\ 4\pi}{(2\pi)^3} \tag{8.1.50}$$

we have:

$$\mathrm{Res}_{k^0=k_-^-}(I_2^{[4]}) = \frac{4\pi}{(2\pi)^4}\int_m^\infty \frac{\mathrm{d}E_k\ E_k\,|\vec{k}|}{p_0(2E_k - i\emptyset^+)(2E_k - p_0 - i\emptyset^+)}. \tag{8.1.51}$$

The residue has a brach cut *inside* the integration region if $p_0 > 2m$. We compute the disc directly using the Cauchy principle value, P:

$$\frac{1}{2E_k - p_0 \pm i\eta} = P\left(\frac{1}{p_0 - 2E_k}\right) \mp i\pi\delta(2E_k - p_0) \tag{8.1.52}$$

$$\Longrightarrow \quad \mathrm{Disc}_{p^2}\Big(\mathrm{Res}_{k^0=k_-^-}(I_2^{[4]})\Big) = \lim_{\eta\to 0}\left(\mathrm{Res}_{k^0=k_-^-}(I_2^{[4]})\Big|_{+i\eta} - \right.$$

$$\left. - \mathrm{Res}_{k^0=k_-^-}(I_2^{[4]})\Big|_{-i\eta}\right) \tag{8.1.53}$$

$$= \frac{2}{(2\pi)^3}\int_m^\infty \frac{\mathrm{d}E_k\ E_k\,|\vec{k}|}{p_0 2E_k}(-2\pi i\delta(p_0 - 2E_k)) \tag{8.1.54}$$

$$= i\frac{\beta}{(4\pi)^2}. \tag{8.1.55}$$

NB. The discontinuity can be computed directly using the replacement:

$$\frac{1}{p_0 - 2E_k + i\emptyset^+} \longrightarrow 2\pi i\delta(p_0 - 2E_k). \tag{8.1.56}$$

For the discontinuity of the full integral we need to sum over all residues inside a closed contour, however it turns out, after closing the contour at $-i\infty$, the residue at k_+^- does not have a branch cut in the integration region (outside $p^0 > 2m$). See figure 8.2.

We can write:

$$\mathrm{Disc}_{p^2}\Big(I_2^{[4]}(p^2, m^2)\Big) = 2\pi i\ \mathrm{Disc}_{p^2}\Big(\mathrm{Res}_{k^0=k_-^-}(I_2^{[4]}(p^2, m^2))\Big) \tag{8.1.57}$$

$$= 2\pi i\left(\frac{i\beta}{16\pi^2}\right) \tag{8.1.58}$$

$$= -\frac{\beta}{8\pi}. \tag{8.1.59}$$

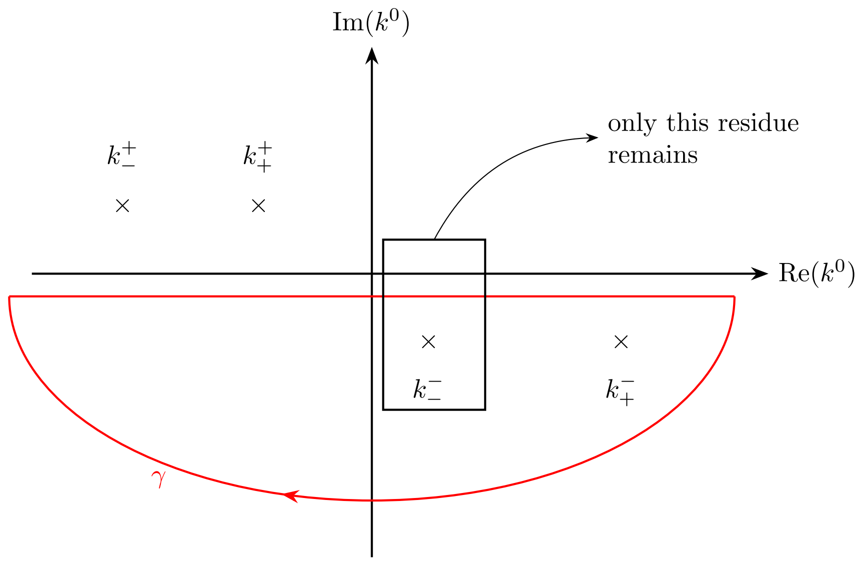


Figure 8.2

Observations. We arrive at the same result by applying th rule:

$$\frac{1}{(k \pm p/2)^2 - m^2 + i\emptyset^+} \longrightarrow -2\pi i\, \delta((k \pm p/2)^2 - m^2)\, \theta(\pm k^0 + p/2) \tag{8.1.60}$$

$$= -2\pi i\, \delta^{(+)}((k \pm p/2)^2 - m^2). \tag{8.1.61}$$

This is the *Cutkosky rule* and can be used to prove the optical thoerem.

Exercise 17. Check, using the closed form for $I_2^{[4-2\epsilon]}(p^2, m^2)$, that the branch cut in $\log(-\beta_+/\beta_-)$ mathches the result obtained above.

We can now continue the analysis of the amplitude since:

$$\mathrm{Disc}_{s_{12}}\Big(\mathcal{A}(p_1p_2 \to p_3p_4)\Big) = -i\frac{\lambda^2}{2}\,\mathrm{Disc}_{s_{12}}\left(\hat{I}_2\left(\frac{s_{12}}{m^2}, \frac{\mu_R^2}{m^2}\right)\right) \tag{8.1.62}$$

$$= -\frac{\lambda^2}{2}\left(-\frac{\beta}{8\pi}\right) \tag{8.1.63}$$

$$= \frac{\lambda^2\beta}{16\pi}. \tag{8.1.64}$$

Since the initial and final states are the same ($\phi\phi \to \phi\phi$), the optical theorem applies.

Exercise 18. Confirm using the final state averaged tree-level amplitude squared:

$$\langle|\mathcal{A}_4^{(0)}|^2\rangle = \frac{1}{2}\lambda^2 \tag{8.1.65}$$

$$\int \mathrm{d}\Phi_2\,\langle|\mathcal{A}_4^{(0)}|^2\rangle = \frac{\lambda^2\beta}{16\pi}. \tag{8.1.66}$$

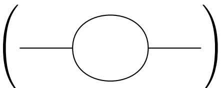
8.2 Generalized unitarity

We saw that the discontinuity of a Feynman integral could be computed using the Cutkosky rules where propagators are replaced with on-shell delta functions:

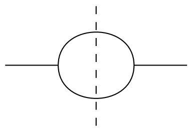
$$\frac{1}{k^2-m^2+i\emptyset^+} \longrightarrow -2\pi i\,\delta^{(+)}(k^2-m^2) \tag{8.2.1}$$

so we can write:

$$\mathrm{Disc}_{p^2}(I_2(p^2,m^2)) = \mathrm{Disc}_{p^2}\left(\right) \tag{8.2.2}$$



$$= \tag{8.2.3}$$



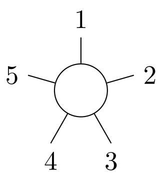
$$= -\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,(2\pi)^2\,\delta(k^2-m^2)\,\delta((k-p)^2-m^2) \tag{8.2.4}$$

$$= -\frac{\beta}{8\pi} \tag{8.2.5}$$

where we call (8.2.3) the *2-particle cut integral*. While we lose connection with unitarity, we can replace any propagator in an amplitude with a on-shell δ-function and obtain a *generalised discontinuity* of the amplitude.

Amplitudes *facturize* when intermediate propagators go on-shell, e.g.:

$$\lim_{s_{12}\to 0} \tag{8.2.6}$$



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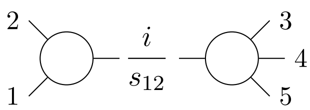


$\frac{i}{s_{12}}$

where the diagrams on the right side are on-shell. This is trivial in scalar theories, but in gauge relies on the spin sum relations:

$$\not{p} + m \overset{p^2 \to m^2}{\longrightarrow} \sum_s u_s \overline{u}_s \tag{8.2.7}$$

$$-g^{\mu\nu} + \frac{p^\mu n^\nu + n^\mu p^\nu}{p \cdot n} \overset{p^2 \to m^2}{\longrightarrow} \sum_n \epsilon_n \epsilon_{-n} \tag{8.2.8}$$

with the *light-like axial gauge*. Hence, when apply cuts to internal lines we may facturize the integrand of any amplitude into products of lower order on-shell amplitudes.

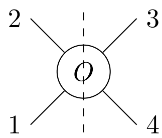
The application of multiple cuts to constrain the form of a scattering amplitude is commonly referred to as *generalized unitarity*.

8.2.1 Double-cuts and dispersion relations

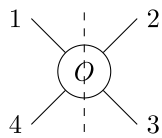
Let's try to use double cut information to construct the 1-loop $\phi\phi \to \phi\phi$ amplitude. We already saw that:

$$\text{Disc}_{s_{12}}(\mathcal{A}_4^{(1)}) = \frac{\lambda^2}{16\pi}\sqrt{1 - \frac{4m^2}{s_{12}}} \tag{8.2.9}$$

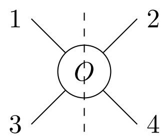
$$= \tag{8.2.10}$$



The cuts in the other channels, s_{14} and s_{13}, are symmetric:



$$= \frac{\lambda^2}{16\pi}\sqrt{1 - \frac{4m^2}{s_{14}}} \tag{8.2.11}$$



$$= \frac{\lambda^2}{16\pi}\sqrt{1 - \frac{4m^2}{s_{13}}}. \tag{8.2.12}$$

To obtain information at the amplitude level we could try to perform dispersion integrals:

$$f(s) = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}s'}{s - s'}\text{Disc}_{s'} f(s'). \tag{8.2.13}$$

Since we have a multivariate function this route in a bit difficult but can be done *summing over all cut contributions*. There is however a simpler ruoute if we identify a *basis* of integrals for the amplitude. The scalar amplitude is trivial, there's no tensor reduction necessary and we can identify from the diagrams by eye that:

$$\mathcal{A}_4^{(1)} = c_{12} I_2(s_{12}, m^2) + c_{23} I_2(s_{23}, m^2) + c_{13} I_2(s_{13}, m^2). \tag{8.2.14}$$

From which we use the cut to project out the *rational coefficients*, e.g.:

$$\mathrm{Cut}_{12}(\mathcal{A}_4^{(1)}) = c_{12} \mathrm{Cut}_{12}(I_2(s_{12,m^2})) \tag{8.2.15}$$

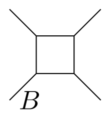
$$\frac{\lambda^2 \beta}{16\pi} = c_{12} \left(-\frac{\beta}{8\pi} \right) \tag{8.2.16}$$

$$\implies \quad c_{12} = -\frac{\lambda^2}{2}. \tag{8.2.17}$$

8.2.2 General one-loop amplitudes

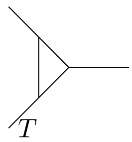
It is possible to show that any one-loop amplitude can, via tensor reduction, be expressed as a linear combination of scalar integrals. If we take external particles to live in $4d$ then the maximum nubers of independent propagators is four, therefore (massless case):

$$\mathcal{A}^{(1)}(p_1, \ldots, p_n) = \sum_{B \in \mathrm{Boxed}} C_B$$



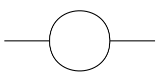
$[1]+$

$$+ \sum_{T \in \mathrm{Triangles}} C_T$$



$[1]+$

$$+ \sum_{b \in \mathrm{Bubbles}} C_b$$



$[1]+$

$$+ R + \mathcal{O}(\epsilon) \tag{8.2.18}$$

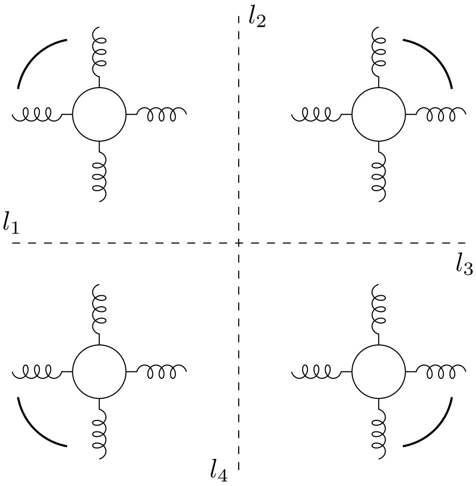
R is a potential rational coefficient coming from:

$$C \cdot I = (C_0 + \epsilon C_1 + \ldots) \left(\frac{I_{-1}}{\epsilon} + I_0 + \ldots \right) \tag{8.2.19}$$

$$= C_0 I + \underbrace{C_1 I_{-1}}_{\epsilon/\epsilon}. \tag{8.2.20}$$

R is of UV origin only and is not seen by cuts in $4d$. By using generalized unitarity cuts we can project out the rational coefficients form the factorised loop integrand. For example:

$$C_B = \frac{1}{2} \sum_{\substack{\text{value where} \\ \lambda_{\text{solution}} \text{of } \phi_i^2 = 0}} \sum_{h_i = \pm}$$



(8.2.21)

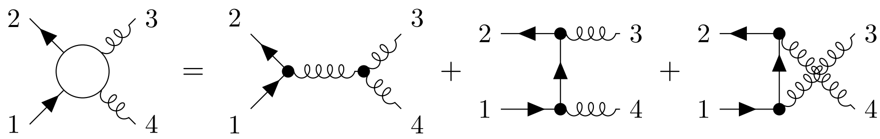
There is no integration to do in the quardupole cut case since the loop momentum is completely fixed.

Observation. Systematic algorithms to proceed iteratively from the *maximal cut* to determine all coefficient and supplying $d = 4 - 2\epsilon$ information to determine R have been developed. Since there is no integration (purely algebraic) this algorithm can be implemented numerically.

8.3 Non-abelian gauge invariance: unitarity and ghost

In chapter §6 we computed the tree-level amplitude for $q\bar{q}gg$ scattering (massless):

$\mathcal{A} =$



(8.3.1)

We showed that the Ward identity was satisfied:

$$\mathcal{A}^\mu p_{3\mu} = 0 \qquad \text{where} \quad \mathcal{A} = \mathcal{A}^\mu \epsilon_{3\mu} \tag{8.3.2}$$

there we will consider a tensor $\mathcal{M}^{\mu\nu}$ where $\mathcal{M}^{\mu\nu}\epsilon_{3\mu}\epsilon_{4\nu} = \mathcal{A}$. $|\mathcal{A}|^2$ is related to $\mathcal{M}$ by the spin sum. We write this is a generic gauge:

$$\sum_s \epsilon_s^\mu(p,n)\epsilon_s^\nu(p,n) = -g^{\mu\nu} + P^{\mu\nu}(p,n) \tag{8.3.3}$$

with:

$$P_{\mu\nu}(p,n) = \frac{p^\mu n^\nu + p^\nu n^\mu}{p \cdot n} \qquad \text{where} \quad n^2 = 0. \tag{8.3.4}$$

Therefore:

$$|\mathcal{A}|^2 = \mathcal{M}^{\mu_3\mu_4}(\mathcal{M}^{\nu_3\nu_4})^* \times \\ \times \Big(- g_{\mu_3\nu_3} + P_{\mu_3\nu_3}(p_3,n)\Big)\Big(- g_{\mu_4\nu_4} + P_{\mu_4\nu_4}(p_4,n)\Big) \tag{8.3.5}$$

naming respectively $1, 2, 3$. The contributions from diagrams 2 and 3 to $\mathcal{M}^{\mu\nu}$ are:

$$\mathcal{M}_{23}^{\mu_3\mu_4} = -ig_s^2 \overline{u}_1 \gamma^{\mu_4} \frac{\not{p}_{23}}{s_{23}} \gamma^{\mu_3} v_2 (t^{a_3} t^{a_4})_{i_2 i_1} - \\ - ig_s^2 \overline{u}_1 \gamma^{\mu_3} \frac{\not{p}_{24}}{s_{24}} \gamma^{\mu_4} v_2 (t^{a_4} t^{a_3})_{i_2 i_1}. \tag{8.3.6}$$

Let's consider the contraction $\mathcal{M}^{\mu_3\mu_4} p_{3\mu}$ (instead of $\mathcal{M}^{\mu_3\mu_4} p_{3\mu_3} \epsilon_{4\mu_4}$ as we did before):

$$\mathcal{M}_{23}^{\mu_3\mu_4} p_{3\mu} = -ig_s^2 \overline{u}_1 \gamma^{\mu_4} \frac{\not{p}_{23}}{s_{23}} \not{p}_3 v_2 (t^{a_3} t^{a_4})_{i_2 i_1} - \\ - ig_s^2 \overline{u}_1 \not{p}_3 \frac{\not{p}_{24}}{s_{24}} \gamma^{\mu_4} v_2 (t^{a_4} t^{a_3})_{i_2 i_1} \tag{8.3.7}$$

$$= -ig_s^2 \overline{u}_1 \gamma^{\mu_4} v_2 [t^{a_3}, t^{a_4}]_{i_2 i_1}. \tag{8.3.8}$$

the contribution from diagram 1 is instead:

$$\mathcal{M}_{1}^{\mu_3\mu_4} p_{3\mu} = ig_s^2 \overline{u}_1 \gamma^\mu v_2 \frac{1}{s_{12}} [t^{a_3}, t^{a_4}]_{i_2 i_1} \Big(p_2^{\mu_4} (p_3 - p_4)_\mu + \\ + g^{\mu_4}{}_\mu 2 p_3 \cdot p_4 + p_{3\mu}(-2p_3 - p_4)^{\mu_4} \Big) \tag{8.3.9}$$

$$= ig_s^2 \overline{u}_1 \gamma^\mu v_2 \frac{1}{s_{12}} \Big(- p_3^{\mu_4} (p_3 + p_4)_\mu + g^{\mu_4}{}_\mu s_{12} - p_{3\mu} p_4^{\mu_4} \Big) [t^{a_3}, t^{a_4}]_{i_2 i_1} \tag{8.3.10}$$

$$= ig_s^2 \Big(\overline{u}_2 \gamma^{\mu_4} v_2 - \overline{u}_1 \not{p}_3 v_2 p_4^{\mu_4} \Big) [t^{a_3}, t^{a_4}]_{i_2 i_1} \tag{8.3.11}$$

so, using the fact that the second term in the right side vanishes when contracted with $\epsilon_{4\mu_4}$, we get:

$$\mathcal{M}^{\mu_3\mu_4} p_{3\mu} = -ig_s^2 \overline{u}_1 \not{p}_3 v_2 p_4^{\mu_4} [t^{a_3}, t^{a_4}]_{i_2 i_1}. \tag{8.3.12}$$

Before jump into a new section let's see some observations. First of all, in QED the color factor is not present, so effectiverly:

$$[t^{a_3}, t^{a_4}]_{i_2 i_1} = 0. \tag{8.3.13}$$

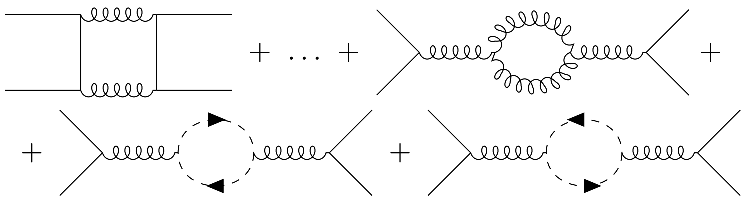
We can also see that what we have found is not in conflict with the Ward identity. Also, there are implications for the optical theorem when connecting branch cuts of loop amplitudes to $|\mathcal{A}|^2$.

The optical theorem states that we must sum over intemediate states:

$$2i\,\mathrm{Im}\{\mathcal{A}(q\bar{q}\to q\bar{q})\} = \sum_k \int \mathrm{d}\Phi_k\,|\mathcal{A}(q\bar{q}\to k)|^2. \tag{8.3.14}$$

In QCD this means we must also consider closed ghost loops and gluon loops on the LHS:

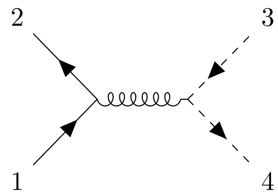
$\mathcal{A}^{(1)}(q\bar{q}\to q\bar{q}) =$



(8.3.15)

To see this is all consistent with our computation of $\mathcal{M}^{\mu_3\mu_4}$ we may also look at another amplitude:

$\mathcal{A}_{ghost}(3,4) =$



(8.3.16)

$$= -ig_s^2\bar{u}_1\gamma^\mu v_2\frac{1}{s_{12}}p_{3\mu}[t^{a_4},t^{a_3}]_{i_2i_1} \tag{8.3.17}$$

$$= -ig_s^2\bar{u}_1\not{p}_3 v_2\frac{1}{s_{12}}[t^{a_4},t^{a_3}]_{i_2i_1}. \tag{8.3.18}$$

This is very similar to the result obtained for:

$$\mathcal{M}^{\mu_3\mu_4}p_{3\mu_3} = p_4^{\mu_4}\mathcal{A}_{ghost}(3,4). \tag{8.3.19}$$

Swapping $3\leftrightarrow 4$ give another relation:

$$\mathcal{M}^{\mu_3\mu_4}p_{4\mu_4} = p_3^{\mu_3}\mathcal{A}_{ghost}(4,3) \tag{8.3.20}$$

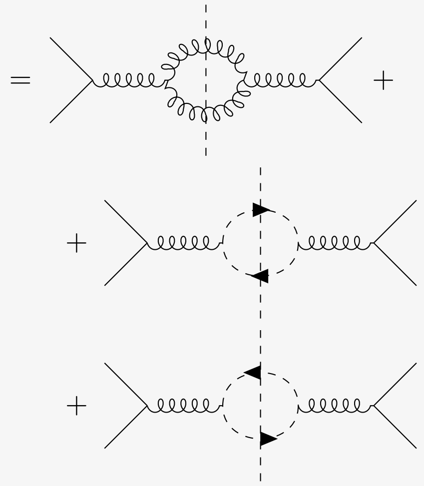
that we can use when expanding the tree amplitude squared using (8.3.5):

$$|\mathcal{A}|^2 = \mathcal{M}^{\mu_3\mu_4}(\mathcal{M}^{\nu_3\nu_4})^*\times \tag{8.3.21}$$

$$\times\Big(-g_{\mu_3\nu_3}+\frac{P_{3\mu_3}n_{\nu_3}+P_{3\nu_3}n_{\mu_3}}{P_3\cdot n}\Big)\Big(-g_{\mu_4\nu_4}+\frac{P_{4\mu_4}n_{\nu_4}+P_{4\nu_4}n_{\mu_4}}{P_4\cdot n}\Big). \tag{8.3.22}$$

Exercise 19. Show that:

$$(8.3.22) = |\mathcal{M}|^2 + |\mathcal{A}_{ghost}(3,4)|^2 + |\mathcal{A}_{ghost}(4,3)|^2$$
$$= \begin{pmatrix}\text{squared amplitude}\\\text{using Feynman gauge}\\\text{propagator}\end{pmatrix} + \begin{pmatrix}\text{ghost loop}\\\text{in one}\\\text{direction}\end{pmatrix} + \begin{pmatrix}\text{ghost loop}\\\text{in opposite}\\\text{direction}\end{pmatrix}$$



So we see that the ghost contributions are essential to mantain unitarity when working in the Feynman gauge. Note that if we work in the *light-like axial gauge*, where the spin sum matches the Feynmn rule for the propagator, unitarity is maintained without needing ghosts.

8.3.1 BRST symmetry

A complete understanding of non-abelian gauge invariance, unitarity and the role of ghost fields may be found using the formalism of *Becchi, Rouet, Stora and Tyutin* (BRST) [′74, ′76].

The issue is that while the classial QCD lagrangian is invariant under $SU(N_c)$ rotation:

$$\mathcal{L}^{cl}_{QCD} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \overline{\psi}(i\not{D} - m)\psi \tag{8.3.23}$$

using 1 quark flavour. Quantization of the gauge field introduces new elements to the lagrangian:

$$\mathcal{L}^{gf}_{QCD} = -\frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 \tag{8.3.24}$$

$$\mathcal{L}^{ghost}_{QCD} = -\overline{c}^a\partial_\mu\mathcal{D}^\mu_{ab}c^b \tag{8.3.25}$$

for (covariant) gauge fixing and ghost respectively. These terms breaks the invariance of the lagrangian under $SU(N_c)$ rotations. There is however a hidden symmetry that we can uncover through the addition of an auxillary scalar field B^a. Let's compare two lagrangians:

$$\mathcal{L}_{QCD} = \mathcal{L}^{cl}_{QCD} + \mathcal{L}^{gf}_{QCD} + \mathcal{L}^{ghost}_{QCD} \tag{8.3.26}$$

$$\mathcal{L}^{B}_{QCD} = \mathcal{L}^{cl}_{QCD} + \mathcal{L}^{ghost}_{QCD} + \frac{\xi}{2}(B^a)^2 + B^a\partial_\mu A^{a\mu} \tag{8.3.27}$$

$$= \mathcal{L}^{cl}_{QCD} + \mathcal{L}^{ghost}_{QCD} + \mathcal{L}^{aux}_{QCD}. \tag{8.3.28}$$

Derivatives of the field B^a do not appear in $\mathcal{L}^{aux}_{QCD}$ hence it is a *non-dynamical field.* We may integrate this field out of the lagrangian using a generating function:

$$\int \mathcal{D}_B \exp\left\{ i \int \mathrm{d}^4x\, \mathcal{L}^{aux}_{QCD} \right\} = \tag{8.3.29}$$

$$= \int \mathcal{D}_B \exp\left\{ i\frac{\xi}{2} \int \mathrm{d}^4x \left(B^a + \frac{1}{\xi}\partial_\mu A^{a\mu} \right)^2 - \frac{1}{\xi^2}(\partial_\mu A^{a\mu})^2 \right\} \tag{8.3.30}$$

$$= \exp\left\{ i \int \mathrm{d}^4x \left(-\frac{1}{2\xi}(\partial_\mu A^{a\mu}) \right) \right\} \int \mathcal{D}_B \exp\left\{ i \int \mathrm{d}^4x\, (B^a)^2 \right\} \tag{8.3.31}$$

$$= \exp\left\{ i \int \mathrm{d}^4x \left(-\frac{1}{2\xi}(\partial_\mu A^{a\mu}) \right) \right\} \mathcal{N} \tag{8.3.32}$$

where we put the normalisation:

$$\mathcal{N} = \int \mathcal{D}_B \exp\left\{ i \int \mathrm{d}^4x\, (B^a)^2 \right\} \tag{8.3.33}$$

that will cancel for any correlation function, so we may say:

$$\mathcal{L}^{B}_{QCD} = \mathcal{L}_{QCD} \tag{8.3.34}$$

since they produce identical physics. Recalling correlation functions in a scalar theory:

$$\langle 0 | T\{\phi_1 \dots \phi_n\} | 0 \rangle = \lim_{T\to\infty} \frac{\int \mathcal{D}_\phi\, \phi_1 \dots \phi_n \exp\{ i \int \mathrm{d}^4x\, \mathcal{L} \}}{\int \mathcal{D}_\phi\, \exp\{ i \int \mathrm{d}^4x\, \mathcal{L} \}} \tag{8.3.35}$$

and noticing that $\mathcal{L}^{B}_{QCD}$ is symmetric under a global tranformation, written using an anti-commuting Grassmann parameter θ:

$$\delta_\theta A^{a\mu} = \theta \mathcal{D}^{ab}_\mu c^b \tag{8.3.36}$$

$$\delta_\theta \psi = i g_s \theta c^a t^a \psi \tag{8.3.37}$$

$$\delta_\theta c^a = -\frac{1}{2} g_s \theta f^{abc} c^b c^c \tag{8.3.38}$$

$$\delta_\theta \overline{c}^a = \theta B^a \tag{8.3.39}$$

$$\delta_\theta B^a = 0. \tag{8.3.40}$$

$\delta A^{a\mu}$ and $\delta\psi$ are equivalent to infinitesimal $SU(N_c)$ rotation through and angle θ_{SU}:

$$\Theta = \theta^a_{SU}(x)t^a \tag{8.3.41}$$
$$= \theta c^a(x)t^a \tag{8.3.42}$$

and, recalling that $SU(N_c)$ leads to:

$$A'_\mu = A_\mu - \mathcal{D}_\mu\Theta + \mathcal{O}(\Theta^2) \tag{8.3.43}$$
$$\psi' = \exp\{ig_s\Theta\}\psi = (1 + ig_s\Theta)\psi + \mathcal{O}(\Theta^2) \tag{8.3.44}$$

we have the correct sign, sign since θ anticommutes with c^a:

$$\theta c^a(x) = -c^a(x)\theta. \tag{8.3.45}$$

We may now show that:

$$\delta_\theta(\mathcal{L}_{QCD}) = \delta_\theta(\mathcal{L}^B_{QCD}) = 0. \tag{8.3.46}$$

We have:

$$\mathcal{L}^B_{QCD} = \mathcal{L}^{cl}_{QCD} + \mathcal{L}^{ghost}_{QCD} + \mathcal{L}^{aux}_{QCD} \tag{8.3.47}$$

and we can se the variation (for $SU(N_c)$ rotation):

$$\delta_\theta(\mathcal{L}^{cl}_{QCD}) = 0 \tag{8.3.48}$$
$$\delta_\theta(\mathcal{L}^{aux}_{QCD}) = \delta_\theta\left(\frac{\xi}{2}(B^a)^2 + B^a\partial_\mu A^{a\mu}\right) \tag{8.3.49}$$
$$= \frac{\xi}{2}\underbrace{\delta_\theta\left[(B^a)^2\right]}_{=0} + \underbrace{\delta_\theta(B^a)}_{=0}\partial_\mu A^{a\mu} + B^a\partial_\mu\delta_\theta(A^{a\mu}) \tag{8.3.50}$$
$$= B^a\partial^\mu(\theta\mathcal{D}^{ab}_\mu c^b) \tag{8.3.51}$$

and also:

$$\delta_\theta(\mathcal{L}_{QCD}^{ghost}) = \delta_\theta(-\overline{c}^a\partial_\mu\mathcal{D}^{\mu ab}c^b) \tag{8.3.52}$$

$$= -\delta_\theta(\overline{c}^a)\partial_\mu\mathcal{D}^{\mu ab}c^b - \overline{c}^a\partial_\mu\delta_\theta\left(\partial^\mu c^a + g_s f^{abc}A^{\mu c}c^b\right) \tag{8.3.53}$$

$$= -\theta B^a\partial_\mu\mathcal{D}^{\mu ab}c^b - \overline{c}^a\partial_\mu\partial^\mu\left(-\frac{1}{2}\theta f^{abc}c^bc^c\right) - $$

$$- \overline{c}^a\partial_\mu\left[g_s f^{abc}\Big((\delta_\theta A^{\mu c})c^b + A^{\mu c}\delta_\theta(c^b)\Big)\right] \tag{8.3.54}$$

$$= -\theta B^a\partial_\mu\mathcal{D}^{\mu ab}c^b - \overline{c}^a\partial_\mu\partial^\mu\left(-\frac{1}{2}g_s\theta f^{abc}c^bc^c\right) - $$

$$- g_s f^{abc}\overline{c}^a\partial_\mu\left[\theta(\mathcal{D}^{\mu cd}c^d)c^b + A^{\mu c}\left(-\frac{1}{2}g_s\theta f^{bde}c^dc^e\right)\right] \tag{8.3.55}$$

$$= -\theta B^a\partial_\mu\mathcal{D}^{\mu ab}c^b + g_s\overline{c}^a\partial_\mu\theta f^{abc}\frac{1}{2}\partial^\mu(c^bc^c) - $$

$$- g_s f^{abc}\overline{c}^a\theta\partial_\mu\left[(\partial^\mu c^c)c^b + g_s f^{cde}A^{\mu e}c^dc^b - \frac{1}{2}g_s f^{bde}A^{\mu c}c^dc^e\right]. \tag{8.3.56}$$

Exercise 20. Collect terms at $\mathcal{O}(g_s)$ and $\mathcal{O}(g_s^2)$ to sohw that $\mathcal{O}(g_s)$ cancel and $\mathcal{O}(g_s^2)$ cancel with help from the Jacobi identity.

This leaves:

$$\delta_\theta(\mathcal{L}_{QCD}^{ghost}) = -\theta B^a\partial_\mu\mathcal{D}^{\mu ab}c^b = -\delta_\theta(\mathcal{L}_{QCD}^{aux}) \tag{8.3.57}$$

and therefore:

$$\delta_\theta(\mathcal{L}_{QCD}) = 0. \tag{8.3.58}$$

This global symmetry implies a conserved (Noether) current:

$$\partial_\mu j^\mu_{BRST} = 0 \tag{8.3.59}$$

from which we may also take a conserved charge:

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_{BRST} = 0 \tag{8.3.60}$$

Solution to Exercise. We can compute:

$$\delta_\theta(\mathcal{L}_{QCD}^{ghost}) = -\theta B^a \partial_\mu \mathcal{D}^{\mu ab} c^b + g_s \bar{c}^a \partial_\mu \theta f^{abc} \frac{1}{2} \partial^\mu (c^b c^c) - \\ - g_s f^{abc} \bar{c}^a \theta \partial_\mu \left[(\partial^\mu c^c) c^b + g_s f^{cde} A^{\mu e} c^d c^b - \frac{1}{2} g_s f^{bde} A^{\mu c} c^d c^e \right] \tag{8.3.61}$$

$$= -\theta B^a \partial_\mu \mathcal{D}^{\mu ab} c^b + \\ + g_s f^{abc} \bar{c}^a \theta \partial_\mu \left[\frac{1}{2} \partial^\mu (c^b c^c) - \partial^\mu (c^c) c^b \right] + \\ + g_s^2 \bar{c}^a \theta \partial_\mu \left(f^{abc} f^{cde} A^{\mu e} c^d c^b - \frac{1}{2} f^{abc} f^{bde} A^{\mu c} c^d c^e \right) \tag{8.3.62}$$

where we can write:

$$g_s f^{abc} \bar{c}^a \theta \partial_\mu \left[\frac{1}{2} \partial^\mu (c^b c^c) - \partial^\mu (c^c) c^b \right] = 0 \tag{8.3.63}$$

because the fields anticommute and f^{abc} are antisymmetric, and we can see:

$$f^{abc} f^{cde} A^{\mu e} c^d c^b - \frac{1}{2} f^{abc} f^{bde} A^{\mu c} c^d c^e \tag{8.3.64}$$

$$= \left(f^{aex} f^{xdc} - \frac{1}{2} f^{axc} f^{xde} \right) A^{\mu c} c^d c^e \tag{8.3.65}$$

$$= \frac{1}{2} \left(f^{aex} f^{xdc} + f^{adx} f^{xec} + f^{acx} f^{xde} \right) A^{\mu c} c^d c^e \tag{8.3.66}$$

$$= 0 \tag{8.3.67}$$

by Jacobi. So:

$$Q_{BRST} = \int \mathrm{d}^3 \vec{x} \, j^0_{BRST} \tag{8.3.68}$$

the BRST charge commutes with the Hamiltonian:

$$[Q_{BRST}, H] = 0. \tag{8.3.69}$$

The variation of the gauge field can be written in terms of the BRST charge:

$$\delta_\theta A^a_\mu = \theta \big[Q_{BRST}, A^a_\mu \big] \tag{8.3.70}$$

and similarly for other fields $(B, c, \bar{c}, \psi)$. We can say this with a simpler notation:

$$\delta_\theta F = \theta Q F. \tag{8.3.71}$$

Exercise 21. Show that $Q^2 F = 0$ for a field:

$$F = (A^a_\mu, \psi, c^a, \bar{c}^a, B^a). \tag{8.3.72}$$

This shows the Q^2 is a *nilpotent* operator and we may write:

$$Q^2 = 0. \tag{8.3.73}$$

8.3.2 implications of BRST symmetry for unitarity

A nilpotent operator that commutes with the hamiltonian will divide the Hilbert space, $\mathcal{H}$, into 3 subspace:

- $\mathcal{H}_0$ with states $|\psi_0\rangle$ where $Q|\psi_0\rangle = 0$ and $\nexists\, |\phi\rangle \in \mathcal{H}$ satisfying $|\psi_0\rangle = Q|\phi\rangle$.
- $\mathcal{H}_1$ with states $|\psi_1\rangle$ where $Q|\psi_1\rangle \neq 0$.
- $\mathcal{H}_2$ with states $|\psi_2\rangle$ where $|\psi_2\rangle = Q|\phi\rangle$ (with $|\phi\rangle \in \mathcal{H}$) such that $Q|\psi_2\rangle = Q^2|\phi\rangle = 0$.

This defines the **BRST cohomology**. While we do not explore this further it is the first hint of a deep mathematical structure.

We can do some observation. We can see that the states of $\mathcal{H}_2$ are orthogonal: for $|\psi_2\rangle, |\psi_2'\rangle \in \mathcal{H}_2$ we can see:

$$\langle\psi_2|\psi_2'\rangle = \langle\phi Q^\dagger Q\phi'\rangle = \langle\phi Q^2\phi'\rangle = 0. \tag{8.3.74}$$

We can see that the states of $\mathcal{H}_2$ are orthogonal to those of $\mathcal{H}_0$:

$$\langle\psi_2|\psi_0\rangle = \langle\psi_2| Q^\dagger |\psi_0\rangle = \langle\psi_2| Q|\psi_0\rangle = 0. \tag{8.3.75}$$

We can interpret the structure as follows:

- $\mathcal{H}_0$ contains physical states.
- $\mathcal{H}_1$ and $\mathcal{H}_2$ contain unphysical states, so the ghosts and longitudinal polarisations.

Using the BRST charge and the states of each subspace we may prove the unitarity of the S-matrix:

$$[Q_{BRST}, H] = 0 \quad \leftrightarrow \quad [Q, S] = 0. \tag{8.3.76}$$

Therefore, for $|\psi_0\rangle$ in $\mathcal{H}_0$:

$$0 = [Q,S]|\psi_0\rangle \tag{8.3.77}$$
$$= (QS - SQ)|\psi_0\rangle \tag{8.3.78}$$
$$= QS|\psi_0\rangle \tag{8.3.79}$$

using $Q|\psi_0\rangle = 0$. This means that the states $S|\psi_0\rangle$ must be in $\mathcal{H}_0$ or in $\mathcal{H}_2$ and not in $\mathcal{H}_1$:

$$S|\psi_0\rangle \in \mathcal{H}_0 \oplus \mathcal{H}_2. \tag{8.3.80}$$

Now we loo at two physical states from $\mathcal{H}_0$, $|\psi_0\rangle$ and $|\psi_0'\rangle$ where:

$$\langle\psi_0'|\psi_0\rangle = \langle\psi_0'| S^\dagger S|\psi_0\rangle \tag{8.3.81}$$

using $S^\dagger S = 1$. We can insert a complete set of states with:

$$\sum_{\psi \in \mathcal{H}} |\psi\rangle \langle\psi| = \mathbb{1} \tag{8.3.82}$$

in this way we get:

$$\langle\psi_0| S^\dagger S |\psi_0\rangle = \sum_{\psi \in \mathcal{H}} \langle\psi_0'| S^\dagger |\psi\rangle \langle\psi| S |\psi_0\rangle \tag{8.3.83}$$

$$\begin{aligned} &= \sum_{\psi'' \in \mathcal{H}_0} \langle\psi_0'| S^\dagger |\psi_0''\rangle \langle\psi_0''| S |\psi_0\rangle + \\ &\quad + \sum_{\psi_1 \in \mathcal{H}_\infty} \langle\psi_0'| S^\dagger |\psi_1\rangle \underline{\langle\psi_1| S |\psi_0\rangle} + \\ &\quad + \sum_{\psi_2 \in \mathcal{H}_\in} \langle\psi_0'| S^\dagger |\psi_2\rangle \underline{\langle\psi_2| S |\psi_0\rangle} \end{aligned} \tag{8.3.84}$$

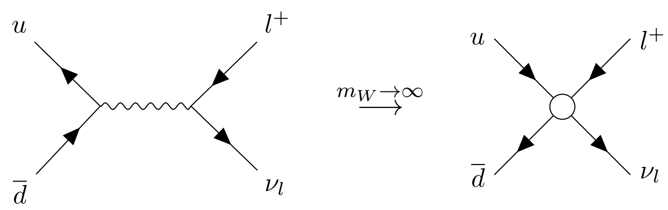
where we can say that the two underlined terms are equals to zero because $\mathcal{H}_1$ is orthogonal to $\mathcal{H}_0 \oplus \mathcal{H}_2$ and $\mathcal{H}_2$ is orthogonal to $\mathcal{H}_1$ and $\mathcal{H}_2$.

So, finally, we can conclude that the S-matrix elements for physical states is unitary.

8.4 Invitation: Effective field theories

We have stated that theories with operators of mass dimension higher than 4 are not renormalizable (in a $4d$ space-time). We may relax this condition if we only require our field theory to apply within a specific energy regime and define an *effective field theory*.

The Fermi theory of the Weak interaction is the most famous example:

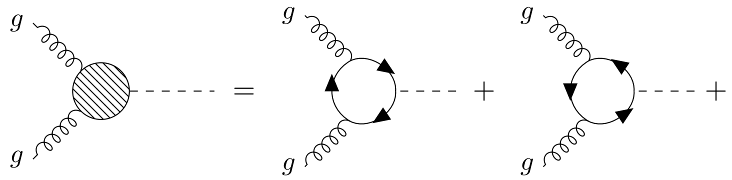


that means that the propagator:

$$\frac{1}{p^2 - m_W^2} \quad \overset{m_W \to \infty}{\longrightarrow} \quad -\frac{1}{m_W^2}\left(1 + \frac{p^2}{m_W^2} + \dots\right) \tag{8.4.1}$$

where the term p^2/m_W^2 is the correction froms higher derivatives suppressed by $1/m_W^2$.

We can use an explicit computation of $gg \to H$ via a top quark loop to illustrate how this can work in practice. The process is loop induced in the Standard Model (SM):



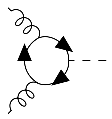
$$+\,\mathcal{O}\left(g_s^4 y_t, g_s^2 y_t^3, g_s^2 y_t g_W^2\right). \qquad (8.4.2)$$

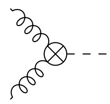
The relevant part of the SM is:

$$\mathcal{L}_{SM} = \ldots\ y_t H \overline{\psi}_t \psi_t + \overline{\psi}_t (i\not{D} - m_t)\psi_t\ \ldots \qquad (8.4.3)$$

distinguishing between L and R handed quark fields is not relevant here. In the region $\sqrt{s} \ll m_t/m_H$ we may use an approximation $m_t \to \infty$ (Higgs couples only the top quark):

$$\begin{array}{ccccc} \mathcal{L}_{SM} & \overset{m_W\to\infty}{\longrightarrow} & \mathcal{L}_{SM}^{(notop)} & + & \mathcal{L}_{eff} \\ \downarrow & & \downarrow & & \downarrow \\ \text{contains} & & \text{no Yukawa} & & \text{contains} \end{array} \qquad (8.4.4)$$



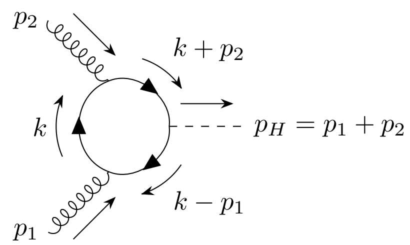


Since we have the technology to compute the loop diagram directly we may see how to write $\mathcal{L}_{eff}$ in terms of a dimension 5 operator. The two diagrams are symmetric so:

$$\mathcal{A}(gg \to H = 2\mathcal{D}) \qquad (8.4.5)$$

where:

$\mathcal{D}$ =



Since the process is loop induced the loop amplitude must be finite. We have:

$$\mathcal{D} = (-ig_s)^2 \left(\frac{-iy_t}{\sqrt{2}}\right) \mathrm{Tr}\{t^{a_1} t^{a_2}\} \epsilon_{1\mu_1} \epsilon_{2\mu_2} I^{\mu_1\mu_2} \qquad (8.4.6)$$

where the loop integral $I^{\mu_1\mu_2}$ is:

$$I^{\mu_1\mu_2} = \int_k \frac{N^{\mu_1\mu_2}}{D_1 D_2 D_3} \qquad (8.4.7)$$

where we have written:

$$D_1 = D(k, m_t) \quad , \quad D_2 = D(k + p_2, m_t) \quad , \quad D_3 = D(k - p_1, m_t). \tag{8.4.8}$$

The numerator is:

$$N^{\mu_1\mu_2} = \mathrm{Tr}\Big\{\not\epsilon_1(\not k + m_t)\not\epsilon_2(\not k + \not p_2 + m_t)(\not k - \not p_1 + m_t)\Big\}. \tag{8.4.9}$$

We will proceed dirctly to Feynman parameters and then perform tensor reduction. Writing:

$$\frac{1}{D_1 D_2 D_3} = 2\int_0^1 \mathrm{d}\alpha_1 \int_0^{1-\alpha_1} \mathrm{d}\alpha_2 \, \frac{1}{\Big[\alpha_1 D_2 + \alpha_2 D_3 + (1 - \alpha_1 - \alpha_2) D_1\Big]^3} \tag{8.4.10}$$

we may show:

$$\hat{D}(k) = \alpha_1 D_2 + \alpha_2 D_3 + (1 - \alpha_1 - \alpha_2) D_1 \tag{8.4.11}$$

$$= (k + \alpha_1 p_2 - \alpha_2 p_1)^2 + 2\alpha_1\alpha_2 p_1 \cdot p_2 - m_t^2 + i\emptyset^+ \tag{8.4.12}$$

where:

$$2p_1 \cdot p_2 = (p_1 + p_2)^2 = p_H^2 = m_H^2. \tag{8.4.13}$$

We may now proceed (dropping the term $i\emptyset^+$) to shift the loop momentum and evaluate the trace in the numerator $N^{\mu_1\mu_2}$.

Exercise 22. Show after Feynman parametrisation the diagram can be written:

$$I^{\mu\nu} = 2\int_k \int_0^1 \mathrm{d}\alpha_1 \int_0^{1-\alpha_1} \mathrm{d}\alpha_2 \, \frac{T^{\mu\nu}(k - \alpha_1 p_2 + \alpha_2 p_1)}{(k^2 - m_t^2 + \alpha_1\alpha_2 m_H^2)^3} \tag{8.4.14}$$

where:

$$\epsilon_{1\mu}\epsilon_{2\nu} T^{\mu\nu}(k) = 4m_t \left[g^{\mu\nu}\left(-k^2 - \frac{m_H^2}{2} - m_t^2\right) + 4k^\mu k^\nu + p_1^\nu p_2^\mu\right] \epsilon_{1\mu}\epsilon_{2\nu}. \tag{8.4.15}$$

You may use $\epsilon_i \cdot p_i = 0$.

For the tensor reduction in Feynman parameter space we make case of:

$$\int_k \frac{k^\mu k^\nu}{(k^2 - \Delta)^n} = \frac{1}{d} g^{\mu\nu} \int_k \frac{k^2}{(k^2 - \Delta)^n} \tag{8.4.16}$$

and:

$$\int \frac{k^\mu}{(k^2 - \Delta)^n} = 0. \tag{8.4.17}$$

Exercise 23. Show after reduction that:

$$I^{\mu\nu} = 2\int_k \int_\alpha \left[g^{\mu\nu}\left(\frac{d-4}{d}k^2 - m_t^2 + m_H^2(1-2\alpha_1\alpha_2)\right) - \right.$$
$$\left. -p_1^\nu p_2^\mu (1-4\alpha_1\alpha_2)\right] \frac{4m_t}{(k^2 - m_t^2 + m_H^2\alpha_1\alpha_2)^3}. \quad (8.4.18)$$

The next step is integration over k using the results:

$$\int_k \frac{1}{(k^2-\Delta)^3} = \frac{i}{(4\pi)^2}(4\pi)^\epsilon \Gamma(1+\epsilon)\frac{\Delta^{-1-\epsilon}}{2} \quad (8.4.19)$$

$$\int_k \frac{k^2}{(k^2-\Delta)^3} = \frac{i}{(4\pi)^2}(4\pi)^\epsilon \Gamma(1+\epsilon)\left(\frac{2-\epsilon}{\epsilon}\right)\Delta^{-\epsilon}. \quad (8.4.20)$$

NB. The $1/\epsilon$ pole in the k^2 integral is cancelled by the coefficient $d-4 = -2\epsilon$ in front so the result is finite as expected.

Exercise 24. Perform loop momentum integration to show (recalling $y_t = m_t/v$):

$$i\mathcal{D} = -\frac{\alpha_s}{\pi}\frac{m_t^2}{v}\frac{\delta^{ab}}{2}\int_0^1 \mathrm{d}\alpha_1 \int_0^{1-\alpha_1} \mathrm{d}\alpha_2 \left(\frac{1-4\alpha_1\alpha_2}{m_t^2 - m_H^2\alpha_1\alpha_2}\right) \times$$
$$\times \left(-\epsilon_1\cdot\epsilon_2 \frac{m_H^2}{2} + \epsilon_1\cdot p_2\, \epsilon_2\cdot p_1\right) + \mathcal{O}(\epsilon). \quad (8.4.21)$$

Exercise 25. Feynman paramater integration. Show that:

$$F\left(\frac{m_H^2}{m_t^2}\right) = \int_\alpha \frac{1-4\alpha_1\alpha_2}{\left(1 - \frac{m_H^2}{m_t^2}\alpha_1\alpha_2\right)^3} \quad (8.4.22)$$

is a constant in the limit $m_t/m_H \to \infty$:

$$\lim_{x\to 0} F(x) = \frac{1}{3} + \mathcal{O}(x). \quad (8.4.23)$$

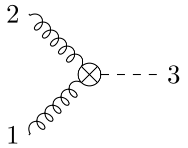
We have now completed the amplitude computation in the $m_t \to \infty$ (or more correctly $(m_t/m_H)^2 \to \infty$) limit:

$$\mathcal{A}(gg\to H) \quad \xrightarrow{\frac{m_t}{m_H}\to\infty} \quad \frac{\alpha_s}{3\pi v}\delta^{ab}(-g^{\mu\nu}p_1\cdot p_2 + p_1^\nu p_2^\mu)\epsilon_{1\mu}\epsilon_{2\nu} + \mathcal{O}\left(\frac{m_H^2}{m_t^2}\right). \quad (8.4.24)$$

The important feature of this result is that it has a local structure that can be matched to a local operator:

$$\mathcal{L}_{eff} = CH\,\mathrm{Tr}\{F^{a}_{\mu\nu}F^{a\mu\nu}\} \tag{8.4.25}$$

where C is known as a *Wilson coefficient*. From this lagrangian we find the Feynman rule:



$$= iC\,(-g^{\mu_1\mu_2}p_1\cdot p_2 + p_1^{\mu_2}p_2^{\mu_1}) \tag{8.4.26}$$

so matching with the result for $\mathcal{A}(gg\to H)$ is straightforward:

$$\begin{array}{ccc} \mathcal{L}_{SM} & & \mathcal{L}_{SM}^{(notop)} + \mathcal{L}_{eff} \\ \downarrow & & \downarrow \\ \mathcal{A}^{(1)}(gg\to H) & \stackrel{m_t/m_H\to\infty}{\longrightarrow} & \mathcal{A}^{(0)}_{eff}(gg\to H) + \mathcal{O}\left(\frac{m_H^2}{m_t^2}\right) \end{array} \tag{8.4.27}$$

$$\Longrightarrow \quad C = \frac{\alpha_s}{3\pi v} + \mathcal{O}(\alpha_s^2). \tag{8.4.28}$$

In general we may construct the set of higher dimension operator satisfying symmetry constraints (e.g. $SU(3)_c \times SU(2)_L \times U(1)_Y$, called $SMEFT$):

$$\mathcal{L}_{eff} = \mathcal{L}_0 + \sum_{k=1}^{\infty}\sum_{m}\frac{C_m^{(k+4)}}{\Lambda^k}O_m^{(k+4)} \tag{8.4.29}$$

where $C_m^{(d)}$ are Wilson coefficient of mass dimension d operators $O_m^{(d)}$. The sum over elements, m, must be done carefully using a basis of independent operators which accounts for field redefinitions, integration-by-parts identifies and equation of motion.

Independent operator basis may be quite large. For the SM, with general flavour structure, we easily reach > 2000 operators.

Appendix

Appendix A

Natural Units

In this section, we study what is meant by natural units. In Physics, it is very common to use *God-given units*, i.e., the so-called natural units. The independent types of quantities are length $[L]$, time $[T]$, mass $[M]$, and — if we also consider thermodynamics and statistical mechanics — temperature $[T]$.

In nature, there are universal dimensional constants, such as the speed of light c, the reduced Planck constant $\hbar$, Newton's gravitational constant G_N, and the Boltzmann constant k. Typically, natural units are defined as those in which we set:

$$\hbar = c = 1 \tag{A.0.1}$$

and consequently, when modifying the quantities (in standard units):

$$\begin{cases} \hbar = 6.66 \cdot 10^{-22} \text{ MeV} \cdot \text{s} \\ c = 3 \cdot 10^{8} \text{ m/s} \\ \hbar c \sim 200 \text{ MeV} \cdot \text{fm} \end{cases} \implies \begin{cases} \hbar = 1 \\ c = 1 \\ \hbar c = 1. \end{cases}$$

Stating that the speed of light is a dimensionless unit quantity implies that the units of length and time are equivalent:

$$[L] = [T] \tag{A.0.2}$$

consequently, we also have:[1]

$$[E] = [\vec{p}] = [M] \tag{A.0.4}$$

For a force F, we have:

$$[F] = [EL^{-1}] = [ML^{-1}] \tag{A.0.5}$$

[1] As a result of the relationship between mass and energy in Special Relativity:

$$E^2 = p^2 + m^2 \tag{A.0.3}$$

therefore, a quantity like $\hbar$, which has the dimensions of an action, i.e., $[ET]$, in natural units is:

$$[\hbar] = [ET] = [ML] = 1 \quad \Longrightarrow \quad [M] = [L^{-1}]. \tag{A.0.6}$$

From the relations observed between the quantities, only one of them is truly independent. Choosing length as the independent one, it holds:

$$[L] = [T] \quad , \quad [E] = [\vec{p}] = [M] = [L^{-1}] \quad , \quad [F] = [L^{-2}]. \tag{A.0.7}$$

Typically, in QFT, cross-sections are expressed in natural units, but this is not very convenient to use in the physical world, as we require standard units. In general (not only for cross-sections), by remembering the units of measurement of $\hbar$ and c, it is possible to convert a quantity from natural units to S.I. units by multiplying by appropriate powers of $\hbar$ and c. We have:

$$[\hbar] = ML^2T^{-1} \qquad ; \qquad [c] = LT^{-1}.$$

Let's look at an example. Take the Thomson cross-section in natural units and convert it to standard units. We have the process:

$$\gamma + e^- \longrightarrow \gamma + e^-$$

with the free e^- and the cross-section:

$$\sigma_T = \frac{8}{3}\pi\frac{\alpha^2}{m_e^2} \tag{A.0.8}$$

which we rewrite as:

$$\sigma_T = \frac{8}{3}\pi\frac{\alpha^2}{m_e^2}\hbar^x c^y \tag{A.0.9}$$

By analyzing the dimensions:

$$\begin{aligned} L^2 &= M^{-2}\left(ML^2T^{-1}\right)^x \left(LT^{-1}\right)^y \\ L^2 &= M^{x-2}L^{2x+y}T^{-x-y}. \end{aligned}$$

We must therefore solve the system:

$$\begin{cases} L:\ 2 = 2x + y \\ M:\ 0 = x - 2 \\ T:\ 0 = -x - y \end{cases} \quad \Longrightarrow \quad \begin{cases} x = -y \\ x = 2 \\ y = -2. \end{cases}$$

Thus, the cross-section is:

$$\sigma_T = \frac{8}{3}\pi\frac{\alpha^2}{m_e^2}\left(\frac{\hbar}{c}\right)^2 = \frac{8\pi}{3}\frac{\alpha^2}{(m_ec^2)^2}(\hbar c)^2 \sim 64\ \text{fm}^2 = 64 \cdot 10^{-2}\ \text{b}. \tag{A.0.10}$$

Appendix B

Project ideas

I collect in this chapter all the project ideas arised during the lectures.

From **Chapter 3 - ϕ^3 theory at one-loop**:

- Renormalization of ϕ^3 in $6d$ at 2-loops ($\overline{MS}$).
- Renormalization of ϕ^3 in $6d$ in MOM scheme (need to compute finite point of $I_3^{(1)[6-2\epsilon]}$).
- Study UV properties of ϕ^4 theory.

From **Chapter 4 - Loop integration methods in dimensional regularization**:

- Study different integral parametrizations, e.g. Baikov, Mellin Barnes etc.
- Study integration-by-parts reduction and differential equations for multi-loop integrals.
- Study graph theory connection to Symanzik polynomials.

From **Chapter 5 - Renormalization of QED at one-loop**:

- Renormalization of related theories e.g. scalar QED.
- Complete renormalization of QED using the on-shell scheme.
- Path integral methods to prove Ward-Takahashi identity.
- Proof of Furry's theorem.

From **Chapter 6 - Renormalization of QCD at one-loop**:

- Compute $\delta_1^{C(1)}$, $\delta_C^{(1)}$ and verify the relation: $\delta_1^{(1)} - \delta_{\psi_g}^{(1)} = \delta_1^{(1)} - \delta_C^{(1)}$.

- Explore the derivation of Ward-Takahashi and Slavnov-Taylor identities using path integral methods.
- Consider $\delta_{g_S}^{(1)}$ in super-symmetric extensions / relatives of QCD ($\mathcal{N} = 1$ / $\mathcal{N} = 4$ super Yang-Mills).
- Use the spinor-helicity method to derive 4-gluon scattering. Explore extensions to higher multiplicity with off-shell recursion relations.

From **Chapter 7 - The renormalization group**:

- Review Wilson's approach to renormalization.
- Describe the parameters of the Standard Model and their renormalization group equations.
- Explore how locality of quantum field theories and ensure quantum fluctuations a short distances may be renormalized.

From **Chapter 8 - Unitarity and gauge invariance**:

- Explore triple cut constraints for vertex function such as:

$$\gamma^* \longrightarrow e^+e^- \tag{B.0.1}$$

in QED (massless limit interesting enough).

- Describe how the backround field method can be used to compute the β function via an effective action.

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