Ward-Takahashi identity in QED and sQED

UNIVERSITY OF TURIN
Faculty of Mathematical, Physical, and Natural Sciences
Master's Degree in Theoretical Physics

Gabriele Cembalo

Academic Year 2025-2026

Table of Contents

1. QED Path integral derivation
   - Renormalization constants
2. Scalar QED
   - Path integral derivation
   - Renormalization constants
   - Counterterms
3. Conclusions
4. Appendix

QED Path integral derivation

The Ward identity and its generalisation by Takahashi are exact relations between 1PI vertex functions and propagators. They follow from the gauge invariance of QED, and play a key role in the proof of the renormalisability of this theory.

The generating functional Z for a system of photons and electrons, given by:

$$\begin{aligned} Z &= N \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(i \int \mathcal{L}_{\text{eff}}\, dx\right) \\ \mathcal{L}_{\text{eff}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi - m\bar{\psi}\psi \\ &\quad - \frac{1}{2\alpha}(\partial^\mu A_\mu)^2 - J^\mu A_\mu + \bar{\eta}\psi + \bar{\psi}\eta. \end{aligned}$$

Due to the gauge fixing term $\mathcal{L}_{\text{eff}}$ is not gauge invariant. The physical consequences of the theory, however, cannot depend on the gauge. So Z *must* be gauge invariant.

QED Path integral derivation

Performing the infinitesimal transformation:

$$\begin{aligned} A_\mu &\longrightarrow A_\mu + \partial_\mu \Lambda, \\ \psi &\longrightarrow \psi - ie\Lambda\psi, \\ \bar{\psi} &\longrightarrow \bar{\psi} + ie\Lambda\bar{\psi} \end{aligned}$$

not all the terms in $\mathcal{L}_{eff}$ are gauge invariant, so the integrand of Z picks up a factor:

$$\exp\left\{ i \int dx \left[-\frac{1}{\alpha}(\partial^\mu A_\mu)\Box\Lambda + J^\mu \partial_\mu \Lambda - ie\Lambda(\bar{\eta}\psi - \bar{\psi}\eta) \right] \right\},$$

which, since Λ is infinitesimal, may be written:

$$1 + i \int dx \left[-\frac{1}{\alpha}\Box(\partial^\mu A_\mu) - \partial^\mu J_\mu - ie(\bar{\eta}\psi - \bar{\psi}\eta) \right] \Lambda(x).$$

QED Path integral derivation

Invariance of Z implies that this operator, when acting on Z, is merely the identity. Since Λ is an arbitrary function, the operator may be written using the substitutions:

$$\psi \to \frac{1}{i}\frac{\delta}{\delta\bar{\eta}}, \qquad \bar{\psi} \to \frac{1}{i}\frac{\delta}{\delta\eta}, \qquad A_\mu \to \frac{1}{i}\frac{\delta}{\delta J^\mu}$$

and we find the functional differential equation:

$$\left[\frac{i}{\alpha}\Box\partial^\mu\frac{\delta}{\delta J^\mu} - \partial^\mu J_\mu - e\left(\bar{\eta}\frac{\delta}{\delta\bar{\eta}} - \eta\frac{\delta}{\delta\eta}\right)\right] Z[\eta,\bar{\eta},J] = 0.$$

Putting $Z = e^{iW}$ we can write:

$$-\frac{\Box\partial^\mu}{\alpha}\frac{\delta W}{\delta J^\mu} - i\partial^\mu J_\mu - ie\left(\bar{\eta}\frac{\delta W}{\delta\bar{\eta}} - \eta\frac{\delta W}{\delta\eta}\right) = 0\ , \quad W = W[\eta,\bar{\eta},J].$$

QED Path integral derivation

We can convert the relation into an equation for the *vertex function* Γ:

$$\Gamma[\psi,\bar{\psi},A_\mu]=W[\eta,\bar{\eta},J_\mu]-\int dx\,(\bar{\eta}\psi+\bar{\psi}\eta+J^\mu A_\mu)$$

which implies that:

$$\begin{aligned}
\frac{\delta\Gamma}{\delta A_\mu(x)} &= -J^\mu(x), & \frac{\delta W}{\delta J_\mu(x)} &= A^\mu(x)\\
\frac{\delta\Gamma}{\delta\psi(x)} &= -\bar{\eta}(x), & \frac{\delta W}{\delta\bar{\eta}(x)} &= \psi(x)\\
\frac{\delta\Gamma}{\delta\bar{\psi}(x)} &= \eta(x), & \frac{\delta W}{\delta\eta(x)} &= \bar{\psi}(x).
\end{aligned}$$

So we have:

$$-\frac{\Box\partial^\mu}{\alpha}A_\mu(x)+i\partial_\mu\frac{\delta\Gamma}{\delta A_\mu(x)}+ie\psi\frac{\delta\Gamma}{\delta\psi(x)}-ie\bar{\psi}\frac{\delta\Gamma}{\delta\bar{\psi}(x)}=0.$$

QED Path integral derivation

Now functionally differentiate this equation twice, with respect to $\bar{\psi}(x_1)$ and $\psi(y_1)$, and put $\bar{\psi} = \psi = A_\mu = 0$. The first term vanishes, and we obtain:

$$i\partial_x^\mu \frac{\delta^3\Gamma[0]}{\delta\bar{\psi}(x_1)\delta\psi(y_1)\delta A^\mu(x)} = ie\delta(x-x_1)\frac{\delta^2\Gamma[0]}{\delta\bar{\psi}(x_1)\delta\psi(y_1)} - \\ - ie\delta(x-y_1)\frac{\delta^2\Gamma[0]}{\delta\bar{\psi}(x_1)\delta\psi(y_1)}.$$

The left-hand side of this equation is the derivative of the (1PI) electron-photon vertex, and the two terms on the right are the inverses of exact propagators.

QED Path integral derivation

We can go to the momentum space. We therefore define the proper vertex function $\Gamma_\mu(p,q,p')$ by:

$$\int dx\, dx_1\, dy_1\, e^{i(p'x_1 - py_1 - qx)} \frac{\delta^3\Gamma[0]}{\delta\bar{\psi}(x_1)\delta\psi(y_1)\delta A^\mu(x)} = \\ = ie(2\pi)^4\delta(p'-p-q)\,\Gamma_\mu(p,q,p').$$

On the other hand, $\delta^2\Gamma/\delta\bar{\psi}\delta\psi$ is the inverse propagator S'_F (we denote S_F the bare propagator), which is:

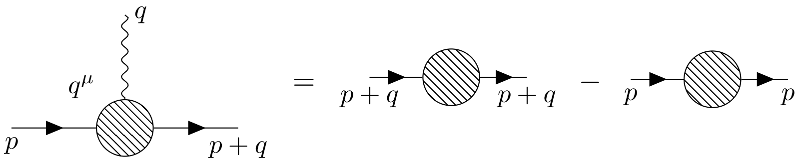
$$\int dx_1\, dy_1\, e^{i(p'x_1 - py_1)} \frac{\delta^2\Gamma[0]}{\delta\bar{\psi}(x_1)\delta\psi(y_1)} = (2\pi)^4\delta(p'-p)\, S_F'^{-1}(p).$$

Multiplying the functional differential equation by $\exp\{i(p'x_1 - py_1 - qx)\}$ and integrating over x, x_1 and y_1 then gives:

$$q^\mu\Gamma_\mu(p,q,p+q) = S_F'^{-1}(p+q) - S_F'^{-1}(p).$$

QED Path integral derivation

This is known as the *Ward-Takahashi identity* and it may be represented pictorially, as:



QED Renormalization constants

Taking the limit $q_\mu \to 0$ (so the fermions go on-shell) yields the *Ward identity*:

$$q^\mu \Gamma_\mu(p,0,p) = q^\mu \frac{\partial S_F'^{-1}}{\partial p^\mu}.$$

This relation holds to all orders in perturbation theory. For instance at the first order (including the corrections from the renormalization process):

$$S_F'^{-1}(p) \sim Z_\psi^{-1}(\gamma_\mu p^\mu - m) = S_F^{-1}(p) \quad \Longrightarrow \quad \frac{\partial S_F'^{-1}(p)}{\partial p^\mu} \sim Z_\psi^{-1}\gamma_\mu$$

$$\Gamma_\mu(p,0,p) \sim Z_1^{-1}\gamma_\mu$$

which verify the Ward identity and show:

$$Z_1^{-1} q^\mu \gamma_\mu = Z_\psi^{-1} q^\mu \frac{\partial S_F^{-1}}{\partial p^\mu} \quad \Longrightarrow \quad Z_1 = Z_\psi.$$

Scalar QED

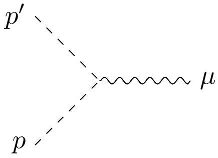
The lagrangian of the theory is:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) - m^2\phi^\dagger\phi \;, \qquad D_\mu = \partial_\mu + ieA_\mu$$

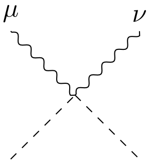
and we have the interaction part:

$$\mathcal{L}_{int} = -ieA^\mu(\phi^\dagger\overleftrightarrow{\partial_\mu}\phi) + e^2A_\mu A^\mu\phi^\dagger\phi$$

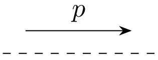
so we have the rules:



$= -ie(p+p')^\mu,$



$= 2ie^2g^{\mu\nu}.$



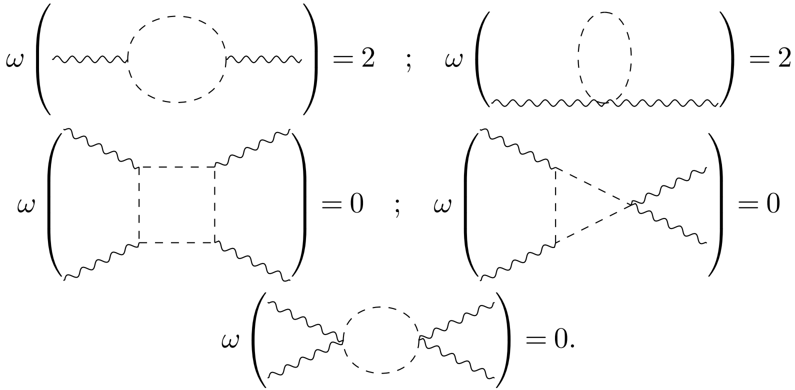
$= \dfrac{i}{p^2-m^2}.$

Scalar QED

The *superficial degree of divergence* ω in $d=4$ dimensions is:

$$\omega = 4L - 2I_B - 2I_\gamma + V_3$$

where: $L =$ loops, $I_{B,\gamma} =$ internal propagators, and $V_3 =$ number of three-point vertices. We have some divergent graph:



sQED Path integral derivation

We can rederive the Ward-Takahashi identity in the Scalar QED following the same step for the QED. We have:

$$Z = N \int \mathcal{D}A_\mu \, \mathcal{D}\phi^\dagger \, \mathcal{D}\phi \, \exp\left(i \int \mathcal{L}_{\text{eff}} \, dx \right),$$

with:

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^\dagger (D^\mu \phi) - m^2 \phi^\dagger \phi - \frac{1}{2\alpha} (\partial^\mu A_\mu)^2 - \\ &\quad - J^\mu A_\mu + j^\dagger \phi + \phi^\dagger j, \\ D_\mu &= \partial_\mu + ieA_\mu. \end{aligned}$$

Due to the gauge-fixing term, $\mathcal{L}_{\text{eff}}$ is not gauge invariant, but we want Z to be.

sQED Path integral derivation

Performing the infinitesimal transformation:

$$A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda, \qquad \phi \longrightarrow \phi - ie\Lambda\phi, \qquad \phi^\dagger \longrightarrow \phi^\dagger + ie\Lambda\phi^\dagger,$$

the covariant kinetic term $(D_\mu\phi)^\dagger(D^\mu\phi)$ and the mass term are invariant by construction; only the gauge-fixing term and the source terms are not. The integrand of Z therefore picks up a factor:

$$\exp\left\{ i\int dx \left[-\frac{1}{\alpha}(\partial^\mu A_\mu)\Box\Lambda - J^\mu\partial_\mu\Lambda - ie\Lambda(j^\dagger\phi - \phi^\dagger j) \right] \right\},$$

which, since Λ is infinitesimal, may be written:

$$1 + i\int dx \left[-\frac{1}{\alpha}\Box(\partial^\mu A_\mu) + \partial^\mu J_\mu - ie(j^\dagger\phi - \phi^\dagger j) \right] \Lambda(x).$$

sQED Path integral derivation

Since Λ is an arbitrary function, the operator may be written using the substitutions:

$$\phi \to \frac{1}{i}\frac{\delta}{\delta j^\dagger}, \qquad \phi^\dagger \to \frac{1}{i}\frac{\delta}{\delta j}, \qquad A_\mu \to \frac{1}{i}\frac{\delta}{\delta J^\mu},$$

giving the functional differential equation:

$$\left[\frac{i}{\alpha}\Box\partial^\mu \frac{\delta}{\delta J^\mu} + \partial^\mu J_\mu - e\left(j^\dagger \frac{\delta}{\delta j^\dagger} - j\frac{\delta}{\delta j}\right)\right] Z[j, j^\dagger, J] = 0.$$

Putting $Z = e^{iW}$:

$$-\frac{\Box\partial^\mu}{\alpha}\frac{\delta W}{\delta J^\mu} + i\partial^\mu J_\mu - ie\left(j^\dagger \frac{\delta W}{\delta j^\dagger} - j\frac{\delta W}{\delta j}\right) = 0$$

sQED Path integral derivation

where $W = W[j, j^\dagger, J]$. Define the vertex functional Γ through the same transformation and relation se get:

$$-\frac{\Box\partial^\mu}{\alpha}A_\mu(x) + i\partial_\mu\frac{\delta\Gamma}{\delta A_\mu(x)} + ie\,\phi(x)\frac{\delta\Gamma}{\delta\phi(x)} - ie\,\phi^\dagger(x)\frac{\delta\Gamma}{\delta\phi^\dagger(x)} = 0.$$

which is identical to the fermionic case with $\phi \leftrightarrow \psi$, $\phi^\dagger \leftrightarrow \bar{\psi}$.
At this point we have to functionally differentiate the equation, but we can do in two ways that bring us to the 3-point vertex and the 4-point vertex. Functionally differentiate this equation twice, with respect to $\phi^\dagger(x_1)$ and $\phi(y_1)$, and put $\phi = \phi^\dagger = A_\mu = 0$. The first term vanishes, and we obtain:

$$i\partial_x^\mu\frac{\delta^3\Gamma[0]}{\delta\phi^\dagger(x_1)\delta\phi(y_1)\delta A^\mu(x)} = ie\,\delta(x - x_1)\frac{\delta^2\Gamma[0]}{\delta\phi^\dagger(x_1)\delta\phi(y_1)} - \\ - ie\,\delta(x - y_1)\frac{\delta^2\Gamma[0]}{\delta\phi^\dagger(x_1)\delta\phi(y_1)}.$$

sQED Path integral derivation

The left-hand side is the derivative of the (1PI) photon-scalar-scalar vertex; the two terms on the right are the inverses of exact propagators. Defining:

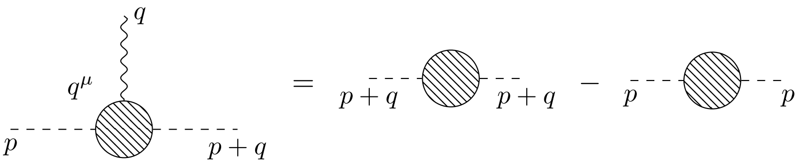
$$\int dx\, dx_1\, dy_1\, e^{i(p' x_1 - p y_1 - q x)} \frac{\delta^3 \Gamma[0]}{\delta\phi^\dagger(x_1)\delta\phi(y_1)\delta A^\mu(x)} = \\ = ie(2\pi)^4 \delta(p' - p - q)\, \Gamma_\mu(p, q, p')$$

$$\int dx_1\, dy_1\, e^{i(p' x_1 - p y_1)} \frac{\delta^2 \Gamma[0]}{\delta\phi^\dagger(x_1)\delta\phi(y_1)} = (2\pi)^4 \delta(p' - p)\, S_B'^{-1}(p)$$

and multiplying the functional derivative equation by $\exp\{i(p' x_1 - p y_1 - q x)\}$ and integrating over x, x_1, y_1 gives:

$$q^\mu\, \Gamma_\mu(p, q, p+q) = S_B'^{-1}(p+q) - S_B'^{-1}(p)$$

which is exactly the same relation as in fermionic QED. Pictorially:



In the soft limit $q^\mu \to 0$ we have:

$$q^\mu \Gamma_\mu(p,0,p) = q^\mu \frac{\partial S'_B(p)}{\partial p^\mu}.$$

sQED Path integral derivation

Now we can functionally differentiate three times, respect to $\phi^\dagger(x_1)$, $\phi(y_1)$ and $A^\nu(z_1)$, and setting $\phi = \phi^\dagger = A_\mu = 0$ we obtain:

$$i\partial_x^\mu \frac{\delta^4\Gamma[0]}{\delta\phi^\dagger(x_1)\delta\phi(y_1)\delta A^\nu(z_1)\delta A^\mu(x)} = ie\,\delta(x-x_1)\frac{\delta^3\Gamma[0]}{\delta\phi^\dagger(x_1)\delta\phi(y_1)\delta A^\nu(z_1)} - \\ - ie\,\delta(x-y_1)\frac{\delta^3\Gamma[0]}{\delta\phi^\dagger(x_1)\delta\phi(y_1)\delta A^\nu(z_1)}.$$

We have to define the four-point vertex:

$$\int dx\,dx_1\,dy_1\,dz_1\,e^{i(p'x_1-py_1-q_1x-q_2z_1)}\frac{\delta^4\Gamma[0]}{\delta\phi^\dagger(x_1)\delta\phi(y_1)\delta A^\nu(z_1)\delta A^\mu(x)} = \\ = ie^2(2\pi)^4\delta(p'-p-q_1-q_2)\,\Gamma_{\mu\nu}(p,q_1,q_2,p'),$$

sQED Path integral derivation

where q_1 and q_2 are the incoming momenta of the two photons carrying Lorentz indices μ and ν respectively. Multiplying the differential equation by the exponential factor and integrating over the spatial coordinates:

$$q_1^\mu \Gamma_{\mu\nu}(p, q_1, q_2, p+q_1+q_2) = \Gamma_\nu(p+q_1, q_2, p+q_1+q_2) - \Gamma_\nu(p, q_2, p+q_2).$$

Expanding for soft photon 1, so $q_1^\mu \to 0$:

$$q_1^\mu \Gamma_{\mu\nu}(p, 0, q_2, p+q_2) = q_1^\mu \frac{\partial \Gamma_\nu(p, q_2, p+q_2)}{\partial p^\mu}.$$

Taking the second soft limit as $q_2 \to 0$, we find:

$$q_1^\mu q_2^\nu \Gamma_{\mu\nu}(p, 0, 0, p) = q_1^\mu q_2^\nu \frac{\partial \Gamma_\nu(p, 0, p)}{\partial p^\mu}$$

substituting the differential Ward-Takahashi identity we conclude:

$$q_1^\mu q_2^\nu \Gamma_{\mu\nu}(p, 0, 0, p) = q_1^\mu q_2^\nu \frac{\partial^2 S_B'^{-1}(p)}{\partial p^\mu \partial p^\nu}.$$

sQED Renormalization constants

As the other theories we must rescale our fields and parameters to absorb the UV divergences:

$$\phi_0 = Z_2^{1/2}\phi_R, \quad A_0^\mu = Z_3^{1/2}A_R^\mu, \quad m_0^2 = m^2 + \delta m^2.$$

The bare interaction vertices are defined through the three-point vertex renormalization constant Z_1 and the four-point (Seagull) vertex renormalization constant Z_4:

$$\Gamma_0^\mu = Z_1^{-1}\Gamma_R^\mu \quad , \quad \Gamma_0^{\mu\nu} = Z_4^{-1}\Gamma_R^{\mu\nu}.$$

We also have:

$$S_B'(p) = \frac{Z_\phi}{p^2 - m^2 - \Sigma(p)} \implies S_B'^{-1}(p) = Z_\phi^{-1}\left(p^2 - m^2 - \Sigma(p)\right)$$

sQED Renormalization constants

So we can see that the Ward identity implies at the first order:

$$\frac{\partial S_B'^{-1}(p)}{\partial p^\mu} \sim 2Z_\phi^{-1} p_\mu \quad , \quad \Gamma_R^\mu(p,q,p') \quad \underset{p'=p}{\xrightarrow{q\to 0}} \quad 2Z_1^{-1} p^\mu$$

hence:

$$2Z_1^{-1} q^\mu p_\mu = 2Z_\phi^{-1} q^\mu p_\mu \quad \Longrightarrow \quad Z_1 = Z_\phi.$$

We can also consider the four-point vertex, where we have two photons with momentum q_1 and q_2. We have:

$$\begin{aligned} \frac{\partial^2 S_B'^{-1}(p)}{\partial p^\mu \partial p^\nu} &= \frac{\partial}{\partial p^\mu}\left[2Z_\phi^{-1} p_\nu\right] \quad , \qquad\qquad \Gamma^{\mu\nu}(p,0,0,p) = Z_4^{-1} 2g^{\mu\nu} \\ &= Z_\phi^{-1} 2g_{\mu\nu} \end{aligned}$$

hence:

$$q_{1\mu} q_{2\nu} Z_4^{-1} 2g^{\mu\nu} = q_{1\mu} q_{2\nu} Z_\phi^{-1} 2g^{\mu\nu} \quad \Longrightarrow \quad Z_4 = Z_\phi.$$

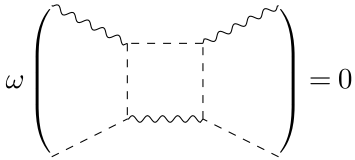
sQED Renormalization constants

So, in sQED we need only *one* renormalization contant:

$$Z_1 = Z_4 = Z_\phi.$$

sQED Counterterms

Having just one renormalization constant is particularly convenient because having diagram like:



we could think that we have to search the relative counterterms in order to absorb the UV divergence. It is therefore sufficient to compute the counterterm for the scalar self-energy: thanks to $Z_1 = Z_4 = Z_\phi$, this automatically fixes the three- and four-point counterterms as well.

Conclusions

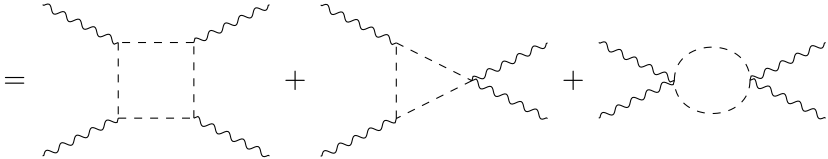
- The Ward-Takahashi identity follows from gauge invariance and holds to all orders in perturbation theory.
- It implies $Z_1 = Z_\psi$ in QED, and $Z_1 = Z_4 = Z_\phi$ in sQED: a single renormalization constant governs all vertices.
- As a result, computing the scalar (or fermion) self-energy counterterm is enough to fix the three- and four-point counterterms automatically.
- Natural extension: the Slavnov-Taylor identities, generalizing this result to non-abelian gauge theories.

Appendix

Scalar QED: $2\gamma \to 2\gamma$

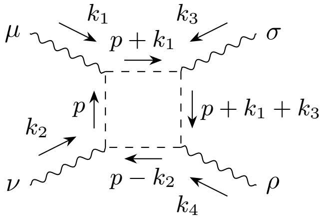
Let's see the process $2\gamma \to 2\gamma$:

$$\mathcal{M} = \mathcal{M}_{\text{Box}} + \mathcal{M}_{\text{Triangle}} + \mathcal{M}_{\text{Bubble}}$$



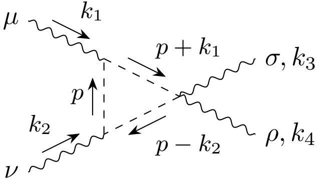
To obtain $\mathcal{M}^{\mu\nu\rho\sigma}$ we need to consider all the possible combinations for the photons attached to the vertices.

Explicitly we have the amplitudes:

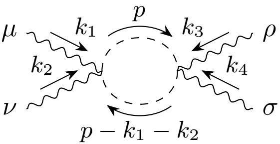


$$\mathcal{M}^{\mu\nu\rho\sigma}_{\text{Box}} = (-ie)^4 \int \frac{d^4p}{(2\pi)^4} \frac{(2p+k_1)^\mu\,(2p+2k_1+k_3)^\rho}{[p^2-m^2]\,[(p+k_1)^2-m^2]} \times$$
$$\times \frac{(2p+2k_1+2k_3+k_4)^\sigma\,(2p-k_2)^\nu}{[(p+k_1+k_3)^2-m^2]\,[(p-k_2)^2-m^2]}$$

Scalar QED: $2\gamma \to 2\gamma$



$$\mathcal{M}^{\mu\nu\rho\sigma}_{\text{Triangle}} = (-ie)^2(2ie^2)\int \frac{d^4p}{(2\pi)^4}\frac{(2p+k_1)^\mu\,(g^{\rho\sigma})\,(2p-k_2)^\nu}{[p^2-m^2]\,[(p+k_1)^2-m^2]\,[(p-k_2)^2-m^2]}$$



$$\mathcal{M}^{\mu\nu\rho\sigma}_{\text{Bubble}} = (2ie^2)^2\int \frac{d^4p}{(2\pi)^4}\frac{g^{\mu\nu}\,g^{\rho\sigma}}{[p^2-m^2]\,[(p-k_1-k_2)^2-m^2]}$$

Scalar QED: $2\gamma \to 2\gamma$ gauge invariance

We want to contract $\mathcal{M}^{\mu\nu\rho\sigma}_{\mathsf{Box}}$ with $k_{1\mu}$; we have:

$$\begin{aligned}
\mathcal{M}^{\mu\nu\rho\sigma}_{\mathsf{Box}} &\propto \frac{1}{p^2-m^2}(2p+k_1)^\mu \frac{1}{(p+k_1)^2-m^2} \\
\Rightarrow \quad k_{1\mu}\mathcal{M}^{\mu\nu\rho\sigma}_{\mathsf{Box}} &\propto \frac{1}{p^2-m^2}\frac{1}{(p+k_1)^2-m^2}(2p\cdot k_1+k_1^2) \\
&= \frac{\left[(p+k_1)^2-m^2\right]-\left[p^2-m^2\right]}{\left[p^2-m^2\right]\left[(p+k_1)^2-m^2\right]}
\end{aligned}$$

So:

$$k_{1\mu}\mathcal{M}^{\mu\nu\rho\sigma}_{\mathsf{Box}} \propto \frac{1}{p^2-m^2} - \frac{1}{(p+k_1)^2-m^2}$$

Scalar QED: $2\gamma \to 2\gamma$ gauge invariance

which is nothing more than the Ward-Takahashi identity for the sQED. Indeed in sQED we have:

$$\Gamma^{\mu}(p+k,p) = (2p+k)^{\mu}$$
$$S^{-1}(p) = p^2 - m^2$$
$$S^{-1}(p+k) = (p+k)^2 - m^2$$

so:

$$k_{\mu}\Gamma^{\mu}(p,p+k) = S^{-1}(p+k) - S^{-1}(p)$$
$$\Longrightarrow k_{\mu}(2p+k)^{\mu} = \left[(p+k)^2 - m^2\right] - \left[p^2 - m^2\right].$$

Scalar QED: $2\gamma \to 2\gamma$ gauge invariance

For the Box diagram we get:

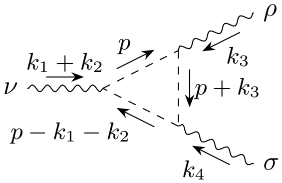
$$k_{1\mu}\mathcal{M}^{\mu\nu\rho\sigma}_{\mathsf{Box}} = e^4 \int \frac{d^4p}{(2\pi)^4} \left\{ \frac{(2p+2k_1+k_3)^\rho\,(2p+2k_1+2k_3+k_4)^\sigma\,(2p-k_2)^\nu}{[p^2-m^2]\,[(p+k_1+k_3)^2-m^2]\,[(p-k_2)^2-m^2]} - \frac{(2p+2k_1+k_3)^\rho\,(2p+2k_1+2k_3+k_4)^\sigma\,(2p-k_2)^\nu}{[(p+k_1)^2-m^2]\,[(p+k_1+k_3)^2-m^2]\,[(p-k_2)^2-m^2]} \right\}.$$

In the second term we can do the shift $p \to p' = p - k_1$, so:

$$k_{1\mu}\mathcal{M}^{\mu\nu\rho\sigma}_{\mathsf{Box}} = e^4 \int \frac{d^4p}{(2\pi)^4} \left\{ \frac{(2p+2k_1+k_3)^\rho\,(2p+2k_1+2k_3+k_4)^\sigma\,(2p-k_2)^\nu}{[p^2-m^2]\,[(p+k_1+k_3)^2-m^2]\,[(p-k_2)^2-m^2]} - \frac{(2p+k_3)^\rho\,(2p+2k_3+k_4)^\sigma\,(2p-2k_1-k_2)^\nu}{[p^2-m^2]\,[(p+k_3)^2-m^2]\,[(p-k_1-k_2)^2-m^2]} \right\}$$

Scalar QED: $2\gamma \to 2\gamma$ gauge invariance

which is quite easy to see that these two integrals are two Triangles with 3 indices. The second term is:



Now, if we consider all the $4! = 24$ combinations of the external photons, then:

$$k_{1\mu}\mathcal{M}^{\mu\nu\rho\sigma}_{\text{Box}} = -\mathcal{M}^{\nu\rho\sigma}_{\text{Triangle}}$$

(see $\mathcal{M}_{\text{Triangle}}$ to verify the sign), which is $\neq 0$, so gauge-variant, but it can help us to verify the gauge invariance of the total $\mathcal{M}$.

Scalar QED: $2\gamma \to 2\gamma$ gauge invariance

Indeed, doing the same thing for the Triangle:

$$
\begin{aligned}
k_{1\mu}\mathcal{M}^{\mu\nu\rho\sigma}_{\text{Triangle}} &\propto \frac{1}{p^2-m^2}(2p+k)^{\mu}k_{1\mu}\frac{1}{[(p+k_1)^2-m^2]} \\
&= \frac{1}{p^2-m^2} - \frac{1}{(p+k_1)^2-m^2}
\end{aligned}
$$

so:

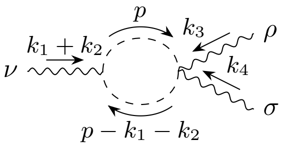
$$
\begin{aligned}
k_{1\mu}\mathcal{M}^{\mu\nu\rho\sigma}_{\text{Triangle}} = -2ie^4 \int \frac{d^4p}{(2\pi)^4} \Bigg\{ &\frac{g^{\rho\sigma}(2p-k_2)^{\nu}}{[p^2-m^2][(p-k_2)^2-m^2]} - \\
&- \frac{g^{\rho\sigma}(2p-k_2)^{\nu}}{[(p+k_1)^2-m^2][(p-k_2)^2-m^2]} \Bigg\}
\end{aligned}
$$

Scalar QED: $2\gamma \to 2\gamma$ gauge invariance

Doing the shift $p \to p' = p - k_1$ in the second term we get:

$$k_{1\mu}\mathcal{M}^{\mu\nu\rho\sigma}_{\text{Triangle}} = -2ie^4 \int \frac{d^4p}{(2\pi)^4} \left\{ \frac{g^{\rho\sigma}(2p-k_2)^\nu}{[p^2-m^2]\,[(p-k_2)^2-m^2]} - \right.$$
$$\left. - \frac{g^{\rho\sigma}(2p-2k_1-k_2)^\nu}{[p^2-m^2]\,[(p-k_1-k_2)^2-m^2]} \right\}$$

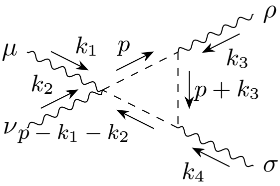
which are two terms with two denominators, so objects like (for example the second term):



which are similar to a Bubble without a photon (external).

Scalar QED: $2\gamma \to 2\gamma$ gauge invariance

Obviously we need to consider all the combinations for the photons, and in this way we can see that a similar contraction $k_{1\mu}\mathcal{M}^{\mu\nu\rho\sigma}_{\text{Triangle}}$ like:



$$\mathcal{M}^{\mu\nu\rho\sigma}_{\text{Triangle}} = \int \frac{d^4p}{(2\pi)^4} \frac{(-ie)^2(2ie^2)\, g^{\mu\nu}(2p+k_3)^\rho(2p-k_4)^\sigma}{[p^2-m^2]\,[(p+k_3)^2-m^2]\,[(p-k_1-k_2)^2-m^2]}$$

where we can see:

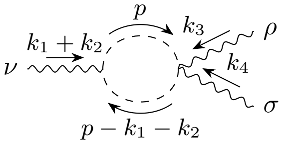
$$k_{1\mu}\mathcal{M}^{\mu\nu\rho\sigma}_{\text{Triangle}} = \int \frac{d^4p}{(2\pi)^4} \frac{(-ie)^2(2ie^2)\, k_1^\nu\,(2p+k_3)^\rho\,(2p-k_4)^\sigma}{[p^2-m^2]\,[(p+k_3)^2-m^2]\,[(p-k_1-k_2)^2-m^2]}$$

Scalar QED: $2\gamma \to 2\gamma$ gauge invariance

which is the "correct" term to cancel the contribution from $k_{1\mu}\mathcal{M}^{\mu\nu\rho\sigma}_{\text{Box}}$.
Let's see the contraction for the Bubble:

$$k_{1\mu}\mathcal{M}^{\mu\nu\rho\sigma}_{\text{Bubble}} = (2ie^2)^2 \int \frac{d^4p}{(2\pi)^4} \frac{k_1^\nu\, g^{\rho\sigma}}{[p^2 - m^2]\,[(p - k_1 - k_2)^2 - m^2]}$$

so a diagram like:



so something similar to the result $k_{1\mu}\mathcal{M}^{\mu\nu\rho\sigma}_{\text{Triangle}}$ (note the sign).

Scalar QED: $2\gamma \to 2\gamma$ gauge invariance

Considering all the photon combinations, we have:

$$\begin{aligned}
k_{1\mu}\mathcal{M}^{\mu\nu\rho\sigma}_{\text{Box}} &= -\mathcal{M}^{\nu\rho\sigma}_{\text{Triangle}} \\
k_{1\mu}\mathcal{M}^{\mu\nu\rho\sigma}_{\text{Triangle}} &= \mathcal{M}^{\nu\rho\sigma}_{\text{Triangle}} - \mathcal{M}^{\nu\rho\sigma}_{\text{Bubble}} \\
k_{1\mu}\mathcal{M}^{\mu\nu\rho\sigma}_{\text{Bubble}} &= \mathcal{M}^{\nu\rho\sigma}_{\text{Bubble}}
\end{aligned}$$

so:

$$\begin{aligned}
k_{1\mu}\mathcal{M}^{\mu\nu\rho\sigma}_{\text{tot}} &= k_{1\mu}\left(\mathcal{M}^{\mu\nu\rho\sigma}_{\text{Box}} + \mathcal{M}^{\mu\nu\rho\sigma}_{\text{Triangle}} + \mathcal{M}^{\mu\nu\rho\sigma}_{\text{Bubble}}\right) \\
&= \left[-\mathcal{M}^{\nu\rho\sigma}_{\text{Triangle}}\right] + \left[\mathcal{M}^{\nu\rho\sigma}_{\text{Triangle}} - \mathcal{M}^{\nu\rho\sigma}_{\text{Bubble}}\right] + \left[\mathcal{M}^{\nu\rho\sigma}_{\text{Bubble}}\right] \\
&= 0.
\end{aligned}$$

So $\mathcal{M}_{\text{tot}}$ is gauge invariant.

Scalar QED: $2\gamma \to 2\gamma$ finite contribution

We've saw that:

$$\omega(\text{Box}) = 0 \quad , \quad \omega(\text{Triangle}) = 0 \quad , \quad \omega(\text{Bubble}) = 0$$

but a logarithmic divergence is necessarily *independent of external momenta*. The divergent part of the amplitude must therefore be a rank-4 tensor constructed exclusively from the metric tensor $g^{\mu\nu}$. The most general form is:

$$\mathcal{M}^{\mu\nu\rho\sigma}_{\text{div}} = C_1\, g^{\mu\nu} g^{\rho\sigma} + C_2\, g^{\mu\rho} g^{\nu\sigma} + C_3\, g^{\mu\sigma} g^{\nu\rho}$$

where C_1, C_2, C_3 are potentially divergent constants.
The gauge invariance of the theory implies the Ward–Takahashi identities. In particular:

$$\begin{aligned} k_{1\mu}\mathcal{M}^{\mu\nu\rho\sigma}_{\text{tot}} &= 0, & \qquad k_{2\nu}\mathcal{M}^{\mu\nu\rho\sigma}_{\text{tot}} &= 0 \\ k_{3\rho}\mathcal{M}^{\mu\nu\rho\sigma}_{\text{tot}} &= 0, & \qquad k_{4\sigma}\mathcal{M}^{\mu\nu\rho\sigma}_{\text{tot}} &= 0. \end{aligned}$$

Scalar QED: $2\gamma \to 2\gamma$ finite contribution

Applying the first of these to the divergent part we have:

$$k_{1\mu}\mathcal{M}^{\mu\nu\rho\sigma}_{\mathsf{div}} = C_1\, k_1^{\nu} g^{\rho\sigma} + C_2\, k_1^{\rho} g^{\nu\sigma} + C_3\, k_1^{\sigma} g^{\nu\rho} = 0.$$

Since this identity must hold for *arbitrary* external momenta and the tensor structures $k_1^{\nu} g^{\rho\sigma}$, $k_1^{\rho} g^{\nu\sigma}$, $k_1^{\sigma} g^{\nu\rho}$ are linearly independent, it necessarily follows that:

$$C_1 = 0, \qquad C_2 = 0, \qquad C_3 = 0.$$

The only rank-4 tensor structure constructed solely from the metric tensor that is compatible with the Ward–Takahashi identities is the vanishing one. Therefore:

$$\boxed{\mathcal{M}^{\mu\nu\rho\sigma}_{\mathsf{div}} = 0.}$$

The UV divergences of the individual diagrams (Box, Triangle, Bubble) cancel exactly in the sum, and the total amplitude $\mathcal{M}^{\mu\nu\rho\sigma}_{\gamma\gamma\to\gamma\gamma}$ is *ultraviolet finite* without the need for a counterterm. This result is a direct consequence of the gauge invariance of the theory.

Bibliography

[1] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to quantum field theory*. Reading, USA: Addison-Wesley, 1995.

[2] L. H. Ryder. *Quantum Field Theory*. Cambridge University Press, 1996.

[3] George F. Sterman. *An Introduction to quantum field theory*. Cambridge University Press, Aug. 1993. ISBN: 978-0-521-31132-8.

[4] J.B. Zuber and C. Itzykson. *Quantum Field Theory*. Dover Books on Physics. Dover Publications, 2012. ISBN: 9780486134697.