USANDO $M^{\mu\nu} = i\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)$ VALUTIAMO $[M^{\mu\nu}, M^{\rho\sigma}]$.

PER NON PERDERE PEZZI SERVIAMOCI DI UNA FUNZIONE $f = f(x^\mu)$;

$$
\begin{aligned}
M^{\mu\nu}M^{\rho\sigma} f &= -1\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)\left(x^\rho\partial^\sigma - x^\sigma\partial^\rho\right) f \\
&= -\left[x^\mu\partial^\nu(x^\rho\partial^\sigma) - x^\mu\partial^\nu(x^\sigma\partial^\rho) - x^\nu\partial^\mu(x^\rho\partial^\sigma) + x^\nu\partial^\mu(x^\sigma\partial^\rho)\right] f(x) \\
&= -\left[x^\mu\left(g^{\nu\rho}\partial^\sigma + x^\rho\partial^\nu\partial^\sigma\right) - x^\mu\left(g^{\nu\sigma}\partial^\rho + x^\sigma\partial^\nu\partial^\rho\right) - \right. \\
&\quad \left. - x^\nu\left(g^{\mu\rho}\partial^\sigma + x^\rho\partial^\mu\partial^\sigma\right) + x^\nu\left(g^{\mu\sigma}\partial^\rho + x^\sigma\partial^\mu\partial^\rho\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\partial^\nu &= g^{\nu\alpha}\partial_\alpha \\
\partial^\nu x^\rho &= g^{\nu\alpha}\partial_\alpha x^\rho \\
&= g^{\nu\alpha}\delta_\alpha{}^\rho \\
&= g^{\nu\rho}
\end{aligned}
$$

IN QUESTO MODO:

$$
[M^{\mu\nu}, M^{\rho\sigma}] f = [M^{\mu\nu}M^{\rho\sigma} - M^{\rho\sigma}M^{\mu\nu}] f =
$$

$$
\begin{aligned}
= -\Big[& \underline{x^\mu g^{\nu\rho}\partial^\sigma} + x^\mu x^\rho\partial^\nu\partial^\sigma - \underline{x^\mu g^{\nu\sigma}\partial^\rho} - x^\mu x^\sigma\partial^\nu\partial^\rho - \underline{x^\nu g^{\mu\rho}\partial^\sigma} - x^\nu x^\rho\partial^\mu\partial^\sigma + \\
&+ \underline{x^\nu g^{\mu\sigma}\partial^\rho} + x^\nu x^\sigma\partial^\mu\partial^\rho\Big] f + \Big[\underline{x^\rho g^{\sigma\mu}\partial^\nu} + x^\rho x^\mu\partial^\sigma\partial^\nu - \underline{x^\rho g^{\sigma\nu}\partial^\mu} - x^\rho x^\nu\partial^\sigma\partial^\mu - \\
&- \underline{x^\sigma g^{\rho\mu}\partial^\nu} - x^\sigma x^\mu\partial^\rho\partial^\nu + \underline{x^\sigma g^{\rho\nu}\partial^\mu} + x^\sigma x^\nu\partial^\rho\partial^\mu\Big] f
\end{aligned}
$$

VISTA LA SIMMETRIA DI $g^{\mu\nu}$

I TERMINI SOTTOLINEATI A ZIG-ZAG SI CANCELLANO, MENTRE GLI ALTRI LI POSSIAMO RACCOGLIERE:

$$
[M^{\mu\nu}, M^{\rho\sigma}] = -\left[g^{\nu\rho}\left(x^\mu\partial^\sigma - x^\sigma\partial^\mu\right) + g^{\nu\sigma}\left(x^\rho\partial^\mu - x^\mu\partial^\rho\right) + \ldots\right.
$$

$$\dots + \underline{g^{\mu\rho}\left(x^\sigma \partial^\nu - x^\nu \partial^\sigma\right)} + g^{\mu\sigma}\underline{\left(x^\nu \partial^\rho - x^\rho \partial^\nu\right)}\Big]$$

POSSIAMO RICONOSCERE : $-i M^{\mu\nu} = \left(x^\mu \partial^\nu - x^\nu \partial^\mu\right)$

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\left[g^{\nu\rho} M^{\mu\sigma} + g^{\nu\sigma} M^{\rho\mu} + g^{\mu\rho} M^{\sigma\nu} + g^{\mu\sigma} M^{\nu\rho}\right]$$

POSSIAMO ANCHE SISTEMARE I SEGNI USANDO LA SIMMETRIA DI $g^{\mu\nu}$ E L'ANTISIMMETRIA DI $M^{\mu\nu}$. LA STRUTTURA DEGLI INDICI DEI TESTI È :

$$[12, 34] = -i\big((14, 23) + (23, 14) - (13, 24) - (24, 13)\big)$$

INDICI M

INDICI g

SISTEMANDO LA NOSTRA SCRITTURA PER OTTENERE :

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\left[(\mu\sigma, \nu\rho) + (\nu\rho, \mu\sigma) - (\mu\rho, \nu\sigma) - (\nu\sigma, \mu\rho)\right]$$

NOI ABBIAMO

$$= -i\left[g^{\mu\sigma} M^{\nu\rho} + g^{\nu\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\nu\sigma} - g^{\nu\sigma} M^{\mu\rho}\right]$$

AVENDO $N_i = \frac{1}{2}\left(J_i + iK_i\right)$ $\qquad N_i^{\dagger} = \frac{1}{2}\left(J_i - iK_i\right)$

E FACILE VERIFICARE

VALGONO:

$[J_i, J_j] = i\varepsilon_{ijk}J_k$

$[K_i, K_j] = -i\varepsilon_{ijk}J_k$

$[J_i, K_j] = i\varepsilon_{ijk}K_k$

$[A,B] = AB - BA$

$[B,A] = BA - AB = -[A,B]$

$$\begin{aligned}
[N_i, N_j^{\dagger}] &= \left[\tfrac{1}{2}(J_i + iK_i)\,,\ \tfrac{1}{2}(J_j - iK_j)\right] \\
&= \tfrac{1}{4}\Big([J_i, J_j] - i[J_i, K_j] + i\underbrace{[K_i, J_j]}_{-[J_j, K_i] = i\varepsilon_{jik}K_k} + [K_i, K_j]\Big) \\
&= \tfrac{1}{4}\Big(i\varepsilon_{ijk}J_k + \varepsilon_{ijk}K_k + i\big(-i\underbrace{\varepsilon_{jik}}_{-\varepsilon_{ijk}}K_k\big) - i\varepsilon_{ijk}J_k\Big) \\
&= \tfrac{1}{4}\Big(i\varepsilon_{ijk}J_k + \varepsilon_{ijk}K_k - \varepsilon_{ijk}K_k - i\varepsilon_{ijk}J_k\Big) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
[N_i, N_j] &= \tfrac{1}{4}[J_i + iK_i\,,\ J_j + iK_j] = \tfrac{1}{4}\Big([J_i, J_j] + i[J_i, K_j] + i\overbrace{[K_i, J_j]}^{-[J_j, K_i]} - [K_i, K_j]\Big) \\
&= \tfrac{1}{4}\Big(i\varepsilon_{ijk}J_k + i(i\varepsilon_{ijk}K_k) + i\big(-i\underbrace{\varepsilon_{jik}}_{-\varepsilon_{ijk}}K_k\big) + i\varepsilon_{ijk}J_k\Big) \\
&= \tfrac{1}{2}i\varepsilon_{ijk}J_k - \tfrac{1}{2}\varepsilon_{ijk}K_k \\
&= i\varepsilon_{ijk}\Big(\tfrac{1}{2}(J_k + iK_k)\Big) = i\varepsilon_{ijk}N_k
\end{aligned}$$

$$[N_i^+, N_j^+] = \frac{1}{4}\left[J_i - iK_i,\ J_j - iK_j\right]$$

$$= \frac{1}{4}\left([J_i, J_j] - i[J_i, K_j] - i\underbrace{[K_i, J_j]}_{-[J_j, K_i] = -i\varepsilon_{jik}K_k = i\varepsilon_{ijk}K_k} - [K_i, K_j]\right)$$

$$= \frac{1}{4}\left(i\varepsilon_{ijk}J_k - i(i\varepsilon_{ijk}K_k) - i(i\varepsilon_{ijk}K_k) + i\varepsilon_{ijk}K_k\right)$$

$$= \frac{1}{2}i\varepsilon_{ijk}J_k + \frac{1}{2}\varepsilon_{ijk}K_k$$

$$= i\varepsilon_{ijk}\frac{1}{2}\left(J_k - iK_k\right)$$

$$= i\varepsilon_{ijk}N_k^+$$

CONOSCO $\quad (M^{\rho\sigma})^{\mu}{}_{\nu} = -i\left(g^{\mu\rho}\delta^{\sigma}{}_{\nu} - g^{\mu\sigma}\delta^{\rho}{}_{\nu}\right)$

POSSIAMO DETERMINARE LE 6 MATRICI $M^{\rho\sigma}$.

$$(M^{01})^{\mu}{}_{\nu} = -i\left(g^{\mu 0}\delta^{1}{}_{\nu} - g^{\mu 1}\delta^{0}{}_{\nu}\right) = -i\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}(0\,1\,0\,0) - \begin{pmatrix}0\\-1\\0\\0\end{pmatrix}(1\,0\,0\,0)\right] = -i\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$$(M^{02})^{\mu}{}_{\nu} = -i\left(g^{\mu 0}\delta^{2}{}_{\nu} - g^{\mu 2}\delta^{0}{}_{\nu}\right) = -i\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}(0\,0\,1\,0) - \begin{pmatrix}0\\0\\-1\\0\end{pmatrix}(1\,0\,0\,0)\right] = -i\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}$$

$$(M^{03})^{\mu}{}_{\nu} = -i\left(g^{\mu 0}\delta^{3}{}_{\nu} - g^{\mu 3}\delta^{0}{}_{\nu}\right) = -i\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}(0\,0\,0\,1) - \begin{pmatrix}0\\0\\0\\-1\end{pmatrix}(1\,0\,0\,0)\right] = -i\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}$$

$$(M^{12})^{\mu}{}_{\nu} = -i\left(g^{\mu 1}\delta^{2}{}_{\nu} - g^{\mu 2}\delta^{1}{}_{\nu}\right) = -i\left[\begin{pmatrix}0\\-1\\0\\0\end{pmatrix}(0\,0\,1\,0) - \begin{pmatrix}0\\0\\-1\\0\end{pmatrix}(0\,1\,0\,0)\right] = -i\begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&+1&0&0\\0&0&0&0\end{pmatrix}$$

$$(M^{23})^{\mu}{}_{\nu} = -i\left(g^{\mu 2}\delta^{3}{}_{\nu} - g^{\mu 3}\delta^{2}{}_{\nu}\right) = -i\left[\begin{pmatrix}0\\0\\-1\\0\end{pmatrix}(0\,0\,0\,1) - \begin{pmatrix}0\\0\\0\\-1\end{pmatrix}(0\,0\,1\,0)\right] = -i\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&+1&0\end{pmatrix}$$

$$(M^{13})^{\mu}{}_{\nu} = -i\left(g^{\mu 1}\delta^{3}{}_{\nu} - g^{\mu 3}\delta^{1}{}_{\nu}\right) = -i\left[\begin{pmatrix}0\\-1\\0\\0\end{pmatrix}(0\,0\,0\,1) - \begin{pmatrix}0\\0\\0\\-1\end{pmatrix}(0\,1\,0\,0)\right] = -i\begin{pmatrix}0&0&0&0\\0&0&0&-1\\0&0&0&0\\0&1&0&0\end{pmatrix}$$

POSSIAMO VERIFICARE CHE (UGUALE A $J_i = \frac{1}{2}\varepsilon_{ijk} M_{jk}$ POICHÈ ABBASSANDO 2 INDICI SPAZIALI ESCE "(-)(-) = +")

$$J_i = \frac{1}{2}\varepsilon_{ijk} M^{jk} \qquad\qquad K^i = M^{0i}$$

È IMMEDIATO

$$K^1 \equiv K_x = M^{01} = -i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K^2 \equiv K_y = M^{02} = -i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K^3 \equiv K_z = M^{03} = -i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

VEDI ILIOPOULOS PAG. 94

$$J^1 \equiv J^x = \frac{1}{2}\varepsilon^{1jk} M^{jk} = \frac{1}{2}\Big(\varepsilon^{123}M^{23} + \underbrace{\varepsilon^{132}M^{32}}_{(-\varepsilon^{123})(-M^{23})}\Big) = \underbrace{\varepsilon^{123}}_{+1} M^{23} = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{pmatrix}$$

$$J^2 \equiv J^y = \frac{1}{2}\varepsilon^{2jk} M^{jk} = \frac{1}{2}\Big(\underbrace{\varepsilon^{213}}_{-\varepsilon^{123}}M^{13} + \underbrace{\varepsilon^{231}M^{31}}_{\varepsilon^{123}(-M^{13})}\Big) = -(M^{13}) = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$J^3 \equiv J^z = \frac{1}{2}\varepsilon^{3jk} M^{jk} = \frac{1}{2}\Big(\underbrace{\varepsilon^{312}M^{12}}_{\varepsilon^{123}M^{12}} + \underbrace{\varepsilon^{321}M^{21}}_{(-\varepsilon^{123})(-M^{12})}\Big) = M^{12} = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

VEDIAMO I MOMENTI ANGOLARI J_i E I BOOST K_i COSA DIVENTANO PER TRASFORMAZIONI DI PARITÀ. $\longrightarrow \Lambda_P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

VEDI §3.7.2 DI SCHWICHTENBERG

VEDIAMO AD ESEMPIO J_1:

$$J_1' = \Lambda_P J_1 \Lambda_P^{+} = \Lambda_P J_1 \Lambda_P^{T}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}^T$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} = J_1$$

ANALOGHI CONTI PER J_2 E J_3. PER IL BOOST K_1:

$$K_1' = \Lambda_P K_1 \Lambda_P^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}^T = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & +i & 0 & 0 \\ +i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -K_1$$

CONTI ANALOGHI PER K_2, K_3.

CON CONTI ANALOGHI SI VEDE PER L'INVERSIONE TEMPORALE:

$$J_i \xrightarrow[T]{} J'_i = \Lambda_T J_i \Lambda_T^T = J_i$$

$$K_i \xrightarrow[T]{} K'_i = \Lambda_T K_i \Lambda_T^T = -K_i$$

QUINDI COMPLESSIVAMENTE SCRIVIAMO

$$J_i \xrightarrow[P]{} J_i \qquad J_i \xrightarrow[T]{} J_i$$

$$K_i \xrightarrow[P]{} -K_i \qquad K_i \xrightarrow[T]{} -K_i$$

$$\Lambda_{L,R} = e^{+\frac{i}{2}\sigma^i(\omega_i \mp i\eta_i)}$$

$$\Lambda^*_{L,R} = e^{-\frac{i}{2}\sigma^{i*}(\omega_i \pm i\eta_i)}$$

$$\Lambda^T_{L,R} = e^{+\frac{i}{2}\sigma^{iT}(\omega_i \mp i\eta_i)}$$

$$\Lambda^{-1}_{L,R} = e^{-\frac{i}{2}\sigma^i(\omega_i \mp i\eta_i)}$$

$$\Lambda^\dagger_{L,R} = e^{-\frac{i}{2}\sigma^{i\dagger}(\omega_i \pm i\eta_i)}$$

$$\begin{cases}(\sigma^2)^* = -\sigma^2 \; ; \; (\sigma^2)^T = -\sigma^2 \\ (\sigma^2)^\dagger = \sigma^2 \; ; \; (\sigma^i)^\dagger = \sigma^i\end{cases}$$

$$\sigma^2\sigma^i\sigma^2 = -\sigma^{i*} \Rightarrow -\sigma^i = \sigma^2\sigma^{i*}\sigma^2$$

(1) $\Lambda^*_{L,R} \xrightarrow{T} \Lambda^\dagger_{L,R} = e^{-\frac{i}{2}\sigma^i(\omega_i \pm i\eta_i)}$

$$\Rightarrow \boxed{\Lambda^\dagger_{L,R} = \Lambda^{-1}_{R,L}}$$

(2) $\boxed{\sigma^2\Lambda^*_{L,R}\sigma^2} = e^{+\frac{i}{2}\sigma^i(\omega_i \pm i\eta_i)} = \boxed{\Lambda_{R,L}}$

$\xRightarrow{\sigma^2 \cdot \sigma^2}$ (3) $\boxed{\Lambda^*_{L,R} = \sigma^2\Lambda_{R,L}\sigma^2}$

(4) (3) $\xrightarrow[\dagger = * + T]{}$ $(\Lambda^*_{L,R})^\dagger = (\sigma^2\Lambda_{R,L}\sigma^2)^\dagger \Rightarrow \Lambda^T_{L,R} = \sigma^2\Lambda^\dagger_{R,L}\sigma^2 \overset{(1)}{\Rightarrow} \boxed{\Lambda^T_{L,R} = \sigma^2\Lambda^{-1}_{L,R}\sigma^2}$

(5) $\boxed{\Lambda^T_L\sigma^2\Lambda_L} = \Lambda^T_L\sigma^2\Lambda_L\sigma^2\sigma^2 \underset{(3)}{=} \Lambda^T_L\Lambda^*_R\sigma^2 \underset{(4)}{=} \sigma^2\Lambda^{-1}_L\sigma^2\Lambda^*_R\sigma^2 \underset{(3)}{=} \sigma^2\Lambda^{-1}_L\Lambda_L = \boxed{\sigma^2}$

CONTO GENERICO PER $\phi(x^\mu)$, $\mathcal{L}(\phi, \partial_\mu \phi)$

GUARDO

$$\Delta S = \int d^4x \, \Delta\mathcal{L} + \int \delta(d^4x)\, \mathcal{L}$$

, SO $\delta(d^4x) = \partial_\mu \delta x^\mu \, d^4x$ PER UNA TRASFORMAZIONE DI COORDINATE (VEDI NOTE)

CALCOLO

$$\begin{aligned}
\Delta\mathcal{L} &= \delta\mathcal{L} + \delta x^\mu \partial_\mu \mathcal{L} \\
&= \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial[\partial_\mu\phi]}\delta(\partial_\mu\phi) + \delta x^\mu \partial_\mu \mathcal{L} \\
&= \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial[\partial_\mu\phi]}\partial_\mu(\delta\phi) + \delta x^\mu \partial_\mu \mathcal{L} \\
&= \underline{\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi} + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\mu\phi]}\delta\phi\right) - \underline{\partial_\mu\frac{\partial\mathcal{L}}{\partial[\partial_\mu\phi]}\delta\phi} + \delta x^\mu \partial_\mu \mathcal{L}
\end{aligned}$$

MI METTO ON-SHELL $\Rightarrow$ EQ MOTO $= 0$

$$\Rightarrow \Delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\mu\phi]}\delta\phi\right) + \delta x^\mu \partial_\mu \mathcal{L}$$

così

$$\Delta S = \int d^4x \left[\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\mu\phi]}\delta\phi\right) + \underbrace{\delta x^\mu \partial_\mu \mathcal{L} + \partial_\mu \delta x^\mu \mathcal{L}}_{\partial_\mu(\delta x^\mu \mathcal{L})}\right]$$

$$\Rightarrow \Delta S = \int d^4x \; \partial_\mu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi]} \delta\phi + \delta x^\mu \mathcal{L} \right)}_{J^\mu} = 0$$

QUINDI ABBIAMO

$$\boxed{J^\mu = \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi]} \delta\phi + \delta x^\mu \mathcal{L}}$$

SE DIFFERENZIO I CASI DI ~~…~~ TRASFORMAZIONE DI COORDINATE:

$x^\mu \to x^\mu + \delta x^\mu \quad \Rightarrow \quad \Delta\phi = 0 \quad \Rightarrow \quad \underline{\delta\phi = -\delta x^\mu \partial_\mu \phi}$

SOSTITUISCO

$$J^\mu = -\frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi]} \delta x^\nu \partial_\nu \phi + \delta x^\mu \mathcal{L} = \left(-\frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi]} \partial_\nu \phi + g^\mu{}_\nu \mathcal{L} \right) \delta x^\nu = \boxed{-T^\mu{}_\nu \delta x^\nu}$$

TENSORE ENERGIA-IMPULSO

VEDIAMO ANCHE IL CASO GENERICO:

$x^\mu \to x^\mu + \delta x^\mu \quad , \quad \phi \to \phi + \alpha \quad \Rightarrow \quad \underline{\Delta\phi = \delta\phi + \delta x^\mu \partial_\mu \phi}$

SOSTITUENDO $\delta\phi = \Delta\phi - \delta x^\mu \partial_\mu \phi$

$$\boxed{J^\mu = -T^\mu{}_\nu \delta x^\nu + \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi]} \Delta\phi}$$

$$\delta S = S'(x) - S(x) = \int d^4x \left(\mathcal{L}'(x) - \mathcal{L}(x) \right) = \int d^4x \, \delta \mathcal{L}$$

$$d^4x' = (1 + \partial_\mu \delta x^\mu) \, d^4x$$

$$\Delta S = S'(x') - S(x) = \int d^4x' \, \mathcal{L}'(x') - \int d^4x \, \mathcal{L}(x) \quad ; \quad \mathcal{L}'(x') = \Delta \mathcal{L} + \mathcal{L}(x)$$

$$= \int (1 + \partial_\mu \delta x^\mu) \left(\Delta \mathcal{L} + \mathcal{L}(x) \right) d^4x - \int d^4x \, \mathcal{L}(x)$$

$$\simeq \int d^4x \left(\Delta \mathcal{L} + \partial_\mu \delta x^\mu \mathcal{L} \right) = \int d^4x \Big(\delta \mathcal{L} + \underbrace{\delta x^\mu \partial_\mu \mathcal{L} + \mathcal{L} \partial_\mu \delta x^\mu}_{\partial_\mu (\delta x^\mu \mathcal{L})} \Big)$$

$$\Delta \mathcal{L} = \delta \mathcal{L} + \delta x^\mu \partial_\mu \mathcal{L}$$

$$\Delta S = \delta S + \int d^4x \, \partial_\mu (\mathcal{L} \delta x^\mu)$$

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DIMOSTRAZIONE $$h = \frac{\vec{J}\cdot\vec{P}}{|\vec{P}|}$$

PARTIAMO DAL SAPERE:

$$W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}$$

$$\begin{cases} W^0 = \frac{1}{2}\varepsilon^{0ijk}P_i M_{jk} = P_i J^i = \vec{P}\cdot\vec{J} \\ \vec{W} = P_0\vec{J} - \vec{P}\times\vec{K} \end{cases}$$

PRENDIAMO $m=0 \Rightarrow P^\mu = (0, \vec{P})$

$$\Rightarrow \begin{cases} P_\mu P^\mu = 0 \\ W_\mu W^\mu = 0 \\ W_\mu P^\mu = 0 \end{cases} \Rightarrow \boxed{W^\mu = h P^\mu}$$

CI METTIAMO IN UN SR CON $P^\mu = (P^0, 0, 0, \pm P^0)$ COSÌ :

RICORDA CHE $m=0 \Rightarrow v=c \Rightarrow E=P^0 \neq 0$
E NON ESISTE UN SR DI RIPOSO

$$W^0 = \vec{P}\cdot\vec{J} = \pm P^0 J^3$$

$$W^i = P^0 J^i \Rightarrow W^3 = \pm P^0 J^3 \Rightarrow \underline{W^3 = W^0}$$

DUNQUE VEDIAMO:

$$W^0 |p\rangle = \underline{\pm P^0 J^3 |p\rangle} = W^3 |p\rangle = \underline{\pm P^0 h |p\rangle}$$

CONFRONTANDO $\Longrightarrow$ $J^3 |p\rangle = h |p\rangle$

QUNQUE GENERALIZZANDO

$$h = \vec{J}\cdot\hat{P} \Rightarrow \boxed{h = \frac{\vec{J}\cdot\vec{P}}{|\vec{P}|}}$$

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi_a]}\delta\phi_a\right) - \partial_\mu \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi_a]}\delta\phi_a$$

$$\phi_a(x) \longrightarrow \phi_a(x) + \delta\phi_a(x)$$

$$\Rightarrow \delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a + \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi_a]}\delta(\partial_\mu \phi_a) = \left(\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi_a]}\right)\delta\phi_a + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi_a]}\delta\phi_a\right)$$

PRENDIAMO $\delta\phi_a$ t.c. $\delta S = 0 \Longrightarrow \delta\mathcal{L} = \partial_\mu F^\mu$

SE ON-SHELL $\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi_a]} = 0 \Longrightarrow \delta\mathcal{L} = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi_a]}\delta\phi_a\right) \equiv \partial_\mu F^\mu$

$\Rightarrow J^\mu = \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi_a]}\delta\phi_a - F^\mu \Rightarrow \partial_\mu J^\mu = 0 \Rightarrow J^\mu$ CONSERVATA

$$\sigma^{0i} = \frac{i}{2}\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix} - \begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right) = \frac{i}{2}\left(\begin{pmatrix}-\sigma^i&0\\0&\sigma^i\end{pmatrix} - \begin{pmatrix}\sigma^i&0\\0&-\sigma^i\end{pmatrix}\right) = i\begin{pmatrix}-\sigma^i&0\\0&\sigma^i\end{pmatrix}$$

$$\sigma^{ij} = \frac{i}{2}\left[\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix}\begin{pmatrix}0&\sigma^j\\-\sigma^j&0\end{pmatrix} - (i\leftrightarrow j)\right]$$

$$= \frac{i}{2}\left[\begin{pmatrix}-\sigma^i\sigma^j&0\\0&-\sigma^i\sigma^j\end{pmatrix} - \begin{pmatrix}-\sigma^j\sigma^i&0\\0&-\sigma^j\sigma^i\end{pmatrix}\right]$$

$$= \frac{i}{2}\begin{pmatrix}[\sigma^j,\sigma^i]&0\\0&[\sigma^j,\sigma^i]\end{pmatrix} = -\varepsilon^{ijk}\begin{pmatrix}\sigma^k&0\\0&\sigma^k\end{pmatrix} = \varepsilon^{ijk}\begin{pmatrix}\sigma^k&0\\0&\sigma^k\end{pmatrix}$$

$$\left([\sigma^i,\sigma^j] = 2i\varepsilon^{ijk}\sigma^k\right)$$

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SAPPIAMO $(\sigma^i)^{2n} = \mathbb{1}$ $\quad (\sigma^i)^{2n+1} = \sigma^i$ $\quad [\sigma^i, \sigma^j] = i\varepsilon^{ijk}\sigma^k$

IN RAPPRESENTAZIONE DI DIRAC $\quad \gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ $\quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$

$$[\gamma^0, \gamma^i] = 2\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

$$\sigma^{0i} = \frac{i}{2}[\gamma^0, \gamma^i] = i\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

$$\begin{cases} M^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu} \\ M^{0i} = K^i = \frac{1}{2}\sigma^{0i} \end{cases}$$

$$\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}^2 = \mathbb{1} \qquad \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

$$\vec{\eta} = \eta\hat{n} = (\omega^{01}, \omega^{02}, \omega^{03})$$

IN GENERALE DATA UNA MATRICE A t.c. $A^{2n} = \mathbb{1}$ $\quad A^{2n+1} = A$

$$\Rightarrow e^{\alpha A} \sim \mathbb{1} + (\alpha A) + \frac{1}{2!}(\alpha)^2\mathbb{1} + \frac{1}{3!}(\alpha)^3 A + \dots$$

$$= \cosh(\alpha)\mathbb{1} + \sinh(\alpha) A$$

NOI PRENDIAMO $\quad A = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$

L'ESPONENTE DI $S(\Lambda)$ È:

$$-\frac{i}{4}\omega_{0i}\sigma^{0i} = -\frac{i}{4}\eta_i \, i\begin{pmatrix}0 & \sigma^i\\ \sigma^i & 0\end{pmatrix} = \frac{\eta}{4}\begin{pmatrix}0 & \vec{\sigma}\cdot\hat{m}\\ \vec{\sigma}\cdot\hat{m} & 0\end{pmatrix} \equiv \frac{\eta}{4}A$$

DUNQUE

$$\exp\left\{-\frac{i}{4}\omega_{0i}\sigma^{0i}\right\} = \mathbb{1}\cosh\left(\frac{\eta}{4}\right) + \sinh\left(\frac{\eta}{4}\right)A$$

$$= \begin{pmatrix}\cosh(\eta/4) & (\vec{\sigma}\cdot\hat{m})\sinh(\eta/4)\\ (\vec{\sigma}\cdot\hat{m})\sinh(\eta/4) & \cosh(\eta/4)\end{pmatrix}$$

IL FATTORE 4 È CONVENZIONE, DEFINENDO

$$\vec{\eta} = \frac{1}{2}(\omega^{01}, \omega^{02}, \omega^{03})$$

SI SAREBBE OTTENUTO COME ARGOMENTO DI cosh E sinh UN FATTORE $\eta/2$

OPPURE È ANCHE COMUNE

$$\sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$$\not{p}\not{p} = \gamma^\mu p_\mu \gamma^\nu p_\nu = \underbrace{\gamma^\mu \gamma^\nu}\, p_\mu p_\nu$$

$$\begin{aligned}
\gamma^\mu \gamma^\nu &= -\gamma^\nu \gamma^\mu + \{\gamma^\mu, \gamma^\nu\} \\
&= -\gamma^\nu \gamma^\mu + 2g^{\mu\nu} \\
&= -(\gamma^\mu \gamma^\nu + [\gamma^\nu, \gamma^\mu]) + 2g^{\mu\nu} \\
&= -\gamma^\mu \gamma^\nu + 2i\sigma^{\nu\mu} + 2g^{\mu\nu}
\end{aligned}$$

$$\Rightarrow 2\gamma^\mu \gamma^\nu = \underbrace{2i\sigma^{\nu\mu}}_{\text{ANTISIMM.}} + \underbrace{2g^{\mu\nu}}_{\text{SIMM.}}$$

$$\Rightarrow \not{p}\not{p} = [\, i\sigma^{\nu\mu} + g^{\mu\nu}\,]\, p_\mu p_\nu = p_\mu p^\mu = p^2$$

$$\langle 0|\phi(x)\phi^\dagger(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-ip(x-y)}$$

(p, θ, φ)

$$p^0(x_0-y_0) - \vec{p}\cdot(\vec{x}-\vec{y}) = p^0(x_0-y_0)$$

$x^\mu - y^\mu > 0$, $\vec{x} = \vec{y} \Rightarrow x^\mu - y^\mu = x^0 - y^0 > 0 \Rightarrow \langle 0|\phi(x)\phi(y)|0\rangle = \displaystyle\int \frac{\vec{p}^2\,dp}{(2\pi)^3}\frac{4\pi}{2E_p} e^{-i\overbrace{p(x-y)}}$

$$= \frac{1}{2\pi^2}\int_0^{+\infty} \frac{dp\,\vec{p}^2}{2\sqrt{\vec{p}^2+m^2}} e^{-i\sqrt{\vec{p}^2+m^2}(x^0-y^0)}$$

$$= \frac{1}{(4\pi^2)}\int_m^{\infty} \frac{dE\,(E^2-m^2)}{\sqrt{E^2-m^2}\,E_p} e^{-iE\underbrace{(x^0-y^0)}_{\equiv t}}$$

$$= \frac{1}{4\pi^2}\int_m^{\infty} dE\,\sqrt{E^2-m^2}\,e^{-iEt} \underset{t\to\infty}{\propto} e^{-imt}$$

$E_p = \sqrt{\vec{p}^2+m^2}$

$E_p^2 - m^2 = \vec{p}^2$

$E_p\,dE_p = dp\cdot p$

$$dp = \frac{E_p}{\sqrt{E^2-m^2}}\,dE$$

$$\langle 0|\phi(x)\phi^\dagger(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-ip(x-y)}$$

$$= \frac{1}{(2\pi)^3}\int_0^\infty \frac{p^2\,dp}{2E_p} e^{-i\vec{p}\cdot\underbrace{(\vec{x}-\vec{y})}_{\vec{z}}} \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\varphi$$

(p, θ, φ)

$$= \frac{2\pi}{(2\pi)^3}\int \frac{p^2\,dp}{E_p} \underbrace{e^{-ipz\cos\theta}\sin\theta\, d\theta}_{\frac{1}{ipz}\frac{d}{d\theta}\left(e^{-ipz\cos\theta}\right)}$$

$$= \frac{1}{(2\pi)^2}\int \frac{p^2\,dp}{E_p} \frac{d\theta}{ipz} \frac{d}{d\theta}\left(e^{-ipz\cos\theta}\right)$$

$$x^0 - y^0 = \qquad \boxed{\Delta(x-y) \underset{t\to\infty}{\sim} e^{-imt}}$$

$$\vec{x} - \vec{y} =$$

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$$\langle 0|\phi(x)\phi^\dagger(y)|0\rangle = \frac{\lambda}{(2\pi)^2}\int \frac{dp}{E_p}\frac{p^2}{ipz}\underbrace{\int_0^\pi d\theta \frac{d}{d\theta}\left(e^{-ipz\cos\theta}\right)}$$

$$= \frac{\lambda}{(2\pi)^2}\int_0^\infty \frac{dp}{E_p}\frac{p}{iz}\left(e^{ipz}-e^{-ipz}\right)$$

$$= \frac{-i}{(2\pi)^2 z}\int_0^{+\infty} dp \frac{p}{\sqrt{p^2+m^2}}\left(e^{ipz}-e^{-ipz}\right) \qquad p \equiv |\vec{p}|$$

$$= \frac{-i}{(2\pi)^2 z}\left\{\int_0^{+\infty} dp \frac{p}{\sqrt{p^2+m^2}} e^{ipz} - \underbrace{\int_0^{+\infty} dp \frac{p}{\sqrt{p^2+m^2}} e^{-ipz}}\right\}$$

$p \to -p$

$$\Rightarrow \int_0^{-\infty}(-dp)\frac{-p}{\sqrt{p^2+m^2}}e^{ipz} = -\int_{-\infty}^0 dp \frac{p}{\sqrt{p^2+m^2}}e^{ipz}$$

$$\boxed{\begin{aligned} &x^0 \; ? \quad \Delta(x-y) \underset{t\to\infty}{\sim} e^{-imt} \\ &? \end{aligned}}$$

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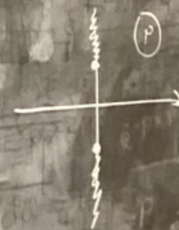
$$\langle 0|\phi(x)\phi^{\dagger}(y)|0\rangle = \frac{-i}{(2\pi)^2 z}\left\{\int_0^{+\infty} dp\, \frac{p}{\sqrt{p^2+m^2}}\, e^{-ipz} + \int_{-\infty}^{0} dp\, \frac{p}{\sqrt{p^2+m^2}}\, e^{ipz}\right\}$$

$$= \frac{-i}{(2\pi)^2 z}\int_{-\infty}^{+\infty} dp\, \frac{p}{\sqrt{p^2+m^2}}\, e^{-ipz}$$

$\longrightarrow$ polidromo $\pm i m$



$x^0 = y^0$

$\vec{x} - \vec{y} = x^\mu - y^\mu$

$\rho = -ip$

$$= 2\,\frac{-i}{(2\pi)^2 z}\int_m^{\infty} d\rho\, \frac{\rho}{\sqrt{m^2-\rho^2}}\, e^{-\rho z} \underset{z\to\infty}{\sim} \boxed{e^{-mz}}$$