Advanced Quantum Field Theory

Exercises — Chapter 4

EXERCISE 2 SHOW THAT WE MAY WRITE THE RANK 2 TENSOR BUBBLE INTEGRAL AS:

$$\underline{I}_2^{(1)[d]\mu\nu} = p^\mu p^\nu \int_0^1 d\alpha\, \alpha^2\, T_2^{[d]}(\Delta) + g^{\mu\nu}\, \frac{1}{d} \int_0^1 d\alpha \left(T_1^{[d]}(\Delta) + \Delta\, T_2^{[d]}(\Delta) \right)$$

WHERE:

$$\Delta = -\alpha(1-\alpha)p^2 + m^2 .$$

YOU CAN ALSO MAY USE THE RESULT:

$$\int_k \frac{k^\mu k^\nu}{(k^2-\Delta)^n} = \frac{1}{d}\, g^{\mu\nu} \int \frac{1}{(k^2-\Delta)^n}$$

SOLUTION WE START WITH THE RANK 2 TENSOR BUBBLE INTEGRAL:

$$\underline{I}_2^{(1)[d]\mu\nu} = \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{D(k,m)\, D(k-p,m)} \qquad \text{WITH} \quad D(k,m) = k^2 - m^2 + i\eta$$

$$= \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2 - m^2 + i\eta)\left((k-p)^2 - m^2 + i\eta\right)}$$

WE CAN USE THE SCHWINGER PARAMETRIZATION:

$$\frac{1}{D^\alpha(k,m)} = \frac{(-1)^m}{\Gamma(\alpha)} \int_0^\infty dt\, t^{\alpha-1}\, e^{-t\, D(k,m)}$$

SO:

$$\underline{I}_2^{(1)[d]\mu\nu} = \frac{(-1)^{2m}}{\Gamma^2(1)} \int \frac{d^d k}{(2\pi)^d} \int_0^\infty dt_1\, dt_2\, \exp\left\{ -t_1\, D(k,m) - t_2\, D(k-p,m) \right\} k^\mu k^\nu$$

WHERE WE USED: $\alpha_1 = \alpha_2 = 1$. WE CAN EXPAND THE EXPRESSION:

$$I_2^{(1)[d]}(p) = \int \frac{d^d k}{(2\pi)^d} \int_0^{+\infty} dt_1 \int_0^{+\infty} dt_2 \exp\left\{-t_1(k^2+m^2) - t_2((k-p)^2+m^2)\right\}$$

WE CAN COMPLETE THE SQUARE IN THE EXPONENTIAL:

$$-t_1(k^2+m^2) - t_2(k^2+p^2-2k\cdot p+m^2) =$$

$$= -k^2(t_1+t_2) + 2t_2 k\cdot p - m^2(t_1+t_2) - t_2 p^2$$

DEFINING $T = t_1 + t_2$

$$= -k^2 T + 2k\cdot p\, t_2 - m^2 T - t_2 p^2$$

$$= -T\left[k^2 - 2\frac{t_2}{T} k\cdot p + m^2 + \frac{t_2}{T} p^2\right]$$

DEFINING $\alpha = t_2/T \Rightarrow t_2 = \alpha T$ AND $t_1 = T - t_2 = T - \alpha T = T(1-\alpha)$

$$= -T\left[k^2 - 2\alpha kp + m^2 + \alpha p^2\right]$$

WE WANT COMPLETE THE SQUARE FOR k:

$$k^2 = (k-\alpha p)^2 - \alpha^2 p^2 + 2\alpha kp \quad \left(\begin{array}{l}\text{SUGGESTED FROM THE}\\ \text{PREVIOUS STEP}\end{array}\right)$$

SO

$$= -T\left[(k-\alpha p)^2 - \alpha^2 p^2 + \alpha p^2 + m^2\right]$$

(2) $$= -T\left[(k-\alpha p)^2 + \alpha(1-\alpha)p^2 + m^2\right]$$

WE CAN REMEMBER OUR DEFINITION:

$$\Delta = -\alpha(1-\alpha)p^2 + m^2$$

SO; WE NEED TO NOTICE

$$dt_1 = \frac{dt_1}{dT}dT = \frac{d(T-t_2)}{dT}dT = dT$$

$$dt_2 = \frac{dt_2}{d\alpha}d\alpha = \frac{d(\alpha T)}{d\alpha}d\alpha = T\,d\alpha$$

TO WRITE:

$$\underline{I}_2^{(1)[d]\mu\nu} = \int \frac{d^dk}{(2\pi)^d} \int_0^1 d\alpha \int_0^{+\infty} dT\, T\, \exp\left\{-T\left[(k-\alpha p)^2 + \Delta\right]\right\} k^\mu k^\nu$$

PERFORMING THE SHIFT $k^\mu \longrightarrow \ell^\mu = k^\mu - \alpha p$ $(dk = d\ell)$ WE HAVE

$$\exp\left\{-T\left[(k-\alpha p)^2 + \Delta\right]\right\} \longrightarrow e^{-T\ell^2} e^{-T\Delta}$$

$$k^\mu k^\nu \longrightarrow (\ell^\mu + \alpha p^\mu)(\ell^\nu + \alpha p^\nu) = \ell^\mu \ell^\nu + \alpha(\ell^\mu p^\nu + \ell^\nu p^\mu) + \alpha^2 p^\mu p^\nu$$

SO THE LOOP

$$\underline{I}_2^{(1)[d]\mu\nu} = \int \frac{d^d\ell}{(2\pi)^d} \int_0^1 d\alpha \int_0^{+\infty} dT\, T\, e^{-T\ell^2} e^{-T\Delta} \left\{\ell^\mu \ell^\nu + \alpha(\ell^\mu p^\nu + \ell^\nu p^\mu) + \alpha^2 p^\mu p^\nu\right\}$$

BUT WE ARE INTEGRATE OVER ℓ, SO INTEGRAL LIKE:

$$\int d\ell \; \ell^\mu \, e^{-T\ell^2} = 0 \qquad \text{(FOR SYMMETRY)}$$

AND WE REMAIN WITH:

$$I_2^{(1)[d]\mu\nu} = \int \frac{d^d\ell}{(2\pi)^d} \int_0^1 d\alpha \int_0^{+\infty} dT \; T \; e^{-T\ell^2} \, e^{-T\Delta} \left\{ \ell^\mu \ell^\nu + \alpha^2 p^\mu p^\nu \right\}$$

BUT WE CAN USE THE RESULT:

$$\int_k k^\mu k^\nu f(k^2) = A \, g^{\mu\nu}$$

$$\Rightarrow \quad A \, g^{\mu\nu} g_{\mu\nu} = Ad \equiv \int_k k^\mu g_{\mu\nu} k^\nu f(k^2)$$

CONTRACTING WITH $g^{\mu\nu}$

$$= \int_k k^2 f(k^2)$$

$$\Rightarrow \quad A = \frac{1}{d} \int k^2 f(k^2)$$

SO WE CAN DO THE STITUTION:

$$k^\mu k^\nu \longrightarrow \frac{1}{d} g^{\mu\nu} \ell^2$$

SO

$$I_2^{(1)[d]\mu\nu} = \int \frac{d^d\ell}{(2\pi)^d} \int_0^1 d\alpha \int_0^{+\infty} dT \; T \; e^{-T\ell} \, e^{-T\Delta} \left\{ \frac{1}{d} g^{\mu\nu} \ell^2 + \alpha^2 p^\mu p^2 \right\}$$

NOW WE CAN SPLIT THE INTEGRAL AND USE THE RESULT FOR GAUSSIAN INTEGRAL!

$$\int \frac{d^d\ell}{(2\pi)^d} e^{-T\ell^2} = \frac{1}{(4\pi)^{d/2}} T^{-d/2}$$

$$\int \frac{d^d\ell}{(2\pi)^d} \ell^2 e^{-T\ell^2} = \frac{d}{2} \frac{1}{(4\pi)^{d/2}} T^{-(\frac{d}{2}+1)}$$

$$I_2^{(1)[d]\mu\nu} = \int_0^1 d\alpha \int_0^{+\infty} dT\ T\ e^{-T\Delta} \left\{ \frac{1}{d} g^{\mu\nu} \frac{d}{2} \frac{1}{(4\pi)^{d/2}} T^{-(\frac{d}{2}+1)} + \alpha^2 p^\mu p^\nu \frac{1}{(4\pi)^{d/2}} T^{-d/2} \right\}$$

$$= \int_0^1 d\alpha \int_0^{+\infty} dT\ T\ e^{-T\Delta} \left\{ \frac{1}{2} g^{\mu\nu} \frac{1}{(4\pi)^{d/2}} T^{-\frac{d}{2}} \frac{1}{T} + \alpha^2 p^\mu p^\nu \frac{1}{(4\pi)^{d/2}} T^{-d/2} \right\}$$

$$= \frac{1}{2} g^{\mu\nu} \int_0^1 d\alpha \int_0^{+\infty} dT\ e^{-T\Delta} \frac{1}{(4\pi)^{d/2}} T^{-d/2} +$$

$$+ p^\mu p^\nu \int_0^1 d\alpha\ \alpha^2 \underbrace{\int_0^{+\infty} dT\ T\ e^{-T\Delta} \frac{1}{(4\pi)^{d/2}} T^{-d/2}}$$

WE CAN NOTICE, DOING $x = T\Delta$ THAT WE HAVE:

$$\frac{1}{(4\pi)^{d/2}} \int_0^{+\infty} dx\ \frac{1}{\Delta^{2-d/2}} x^{1-d/2} e^{-x} = \Delta^{\frac{d}{2}-2} \frac{1}{(4\pi)^{d/2}} \int dx\ x^{(2-\frac{d}{2})-1} e^{-x}$$

$$= \frac{1}{(4\pi)^{d/2}} \Delta^{d/2-2} \Gamma(2-d/2)$$

AND REMEMBERING $T_m^{[d]}(\Delta) = \frac{i(-1)^m}{(4\pi)^{d/2}} \Delta^{d/2-m} \frac{\Gamma(m-d/2)}{\Gamma(m)}$ WE HAVE $T_2^{[d]}(\Delta)$

SO

$$\underline{\underline{I}}_2^{(1)[d]\mu\nu} = \underbrace{\frac{1}{2} g^{\mu\nu} \int_0^1 d\alpha \int_0^{+\infty} dT\, T^{-d/2}\, e^{-T\Delta} \frac{1}{(4\pi)^{d/2}}} + p^\mu p^\nu \int_0^1 d\alpha\, \alpha^2\, T_2^{[d]}(\Delta)$$

IT'S EASYER IF WE DO A STEP BACK TO: (PAG. 4)

$$\frac{1}{d} g^{\mu\nu} \int_0^1 d\alpha \int_0^{+\infty} dT\, T\, e^{-T\Delta} \int \frac{d^d\ell}{(2\pi)^d}\, \ell^2$$

AND WE CAN USE THE FACT:

$$I_2^{(1)[d]} = \int_k \frac{1}{D(k,m)\,D(k-p,m)} = \int_0^1 d\alpha \int_k \frac{1}{(k^2-\Delta)^2}$$

USING THE FEYNMAN TRICK

$$\Rightarrow \int_0^{+\infty} dT\, T\, e^{-T\Delta} \equiv \frac{1}{(k^2-\Delta)^2}$$

AND WE HAVE:

$$\frac{1}{d} g^{\mu\nu} \int_0^1 d\alpha \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2-\Delta)^2} = \frac{1}{d} g^{\mu\nu} \int_0^1 d\alpha \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2-\Delta+\Delta}{(\ell^2-\Delta)^2}$$

$$= \frac{1}{d} g^{\mu\nu} \int_0^1 d\alpha \left\{ \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta)} + \Delta \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta)^2} \right\}$$

SO WITH THE DEFINITION OF THE TADPOLE:

$$T_m^{[d]}(\Delta) = \int_k \frac{1}{(k^2-\Delta)^m}$$

WE CAN CONCLUDE:

$$\underline{\underline{I}}_2^{(1)[d]\mu\nu} = \frac{1}{d} g^{\mu\nu} \int_0^1 d\alpha \left(T_1^{[d]}(\Delta) + \Delta\, T_2^{[d]}(\Delta) \right) + p^\mu p^\nu \int_0^1 d\alpha\, \alpha^2\, T_2^{[d]}(\Delta) .$$

EXERCISE 3 WRITE DOWN THE GENERAL TENSOR DECOMPOSITION FOR A RANK-4 TENSOR TADPOLE:

$$T_m^{[d]\mu_1\mu_2\mu_3\mu_4}(\Delta) = \int_K \frac{k^{\mu_1} k^{\mu_2} k^{\mu_3} k^{\mu_4}}{(k^2-\Delta)^m} .$$

SOLUTION WE START FROM THE DEFINITION:

$$T_m^{[d]\mu_1\mu_2\mu_3\mu_4}(\Delta) = \int \frac{d^d k}{(2\pi)^d} \frac{k^{\mu_1} k^{\mu_2} k^{\mu_3} k^{\mu_4}}{(k^2-\Delta)^m}$$

NOT HAVING EXTERNAL MOMENTUM THE ONLY CHOICE WE CAN DO IS TO BUILD THE RESULT WITH THE METRIC. WE HAVE TO CONSIDER ALL THE PERMUTATION OF THE INDEX:

$$T_m^{[d]\mu_1\mu_2\mu_3\mu_4}(\Delta) = A\left(g^{\mu_1\mu_2}g^{\mu_3\mu_4} + g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_2\mu_3}\right)$$

AS THE OTHER CASE WE CAN CONTRACT THE TENSOR WITH METRIC (INVERSE) $g_{\mu_1\mu_2}\, g_{\mu_3\mu_4}$:

$$\int \frac{d^d k}{(2\pi)^d} g_{\mu_1\mu_2} g_{\mu_3\mu_4} \frac{k^{\mu_1} k^{\mu_2} k^{\mu_3} k^{\mu_4}}{(k^2-\Delta)^m} = A\, g_{\mu_1\mu_2} g_{\mu_3\mu_4} (\dots)$$

$$\Rightarrow \int \frac{d^d k}{(2\pi)^d} \frac{k^4}{(k^2-\Delta)^m} = A\left(g_{\mu_1\mu_2}g^{\mu_1\mu_2}g_{\mu_3\mu_4}g^{\mu_3\mu_4} + g_{\mu_1\mu_2}g^{\mu_1\mu_3}g^{\mu_2\mu_4}g_{\mu_3\mu_4} + g_{\mu_1\mu_2}g^{\mu_1\mu_4}g^{\mu_2\mu_3}g_{\mu_3\mu_4}\right)$$

$$= A\left(d^2 + d + d\right)$$

$$\Rightarrow \int \frac{d^d k}{(2\pi)^d} \frac{k^4}{(k^2-\Delta)^m} = A(d^2+2d)$$

$$\Rightarrow A = \frac{1}{d(d+2)} \int \frac{d^d k}{(2\pi)^d} \frac{k^4}{(k^2-\Delta)^m}$$

SO WE FIND OUT :

$$\overline{T}_m^{[d]\mu_1\mu_2\mu_3\mu_4}(\Delta) = \frac{g^{\mu_1\mu_2}g^{\mu_3\mu_4} + g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_2\mu_3}}{d(d+2)} \int \frac{d^d k}{(2\pi)^d} \frac{k^4}{(k^2-\Delta)^m}$$

BUT WE CAN NOTICE :

$$\frac{k^4}{(k^2-\Delta)^m} = \frac{(k^2)^2}{(k^2-\Delta)^m} = \frac{(k^2-\Delta+\Delta)^2}{(k^2-\Delta)^m} = \frac{(k^2-\Delta)^2 + \Delta^2 + 2\Delta(k^2-\Delta)}{(k^2-\Delta)^m}$$

$$= \frac{1}{(k^2-\Delta)^{m-2}} + \Delta^2 \frac{1}{(k^2-\Delta)^m} + 2\Delta \frac{1}{(k^2-\Delta)^{m-1}}$$

SO USING THE DEFINITION OF TADPOLE : $\overline{T}_m^{[d]}(\Delta) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2-\Delta)^m}$ WE CAN CONCLUDE

$$\overline{T}_m^{[d]\mu_1\mu_2\mu_3\mu_4}(\Delta) = \frac{g^{\mu_1\mu_2}g^{\mu_3\mu_4} + g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_2\mu_3}}{d(d+2)} \left\{ \Delta^2 T_m^{[d]}(\Delta) + 2\Delta T_{m-1}^{[d]}(\Delta) + T_{m-2}^{[d]}(\Delta) \right\}$$

EXERCISE 4 SHOW THAT A GENERAL L-LOOP FEYNMAN INTEGRAL WITH m INTERNAL LINES MAY BE WRITTEN AS:

$$I_m^{(L)[d]}(\nu_1 - \nu_m) = \int \prod_{\ell=1}^{L} \frac{d^d k_\ell}{(2\pi)^d} \frac{1}{\prod_{j=1}^{m} D(q_j, m)^{\nu_j}}$$

$$= \left(\frac{i}{(4\pi)^{d/2}}\right)^L \frac{\Gamma\left(\sum_{i=1}^{m} \nu_i - L\frac{d}{2}\right)}{\prod_{i=1}^{m} \Gamma(\nu_i)} \int_{\alpha_i} \delta\left(1 - \sum \alpha_i\right) \frac{\mathcal{U}^{m-(L+1)\frac{d}{2}}}{(-\mathcal{F})^{m-L\frac{d}{2}}}$$

WHERE $\mathcal{U}$ AND $\mathcal{F}$ ARE THE SYMANZIK POLYNOMIALS.

SOLUTION ESSENTIALLY WE HAVE TO GENERALIZE WHAT WE HAD SAW FOR 2 LOOP. WE CAN USE THE DEFINITION:

$$I_m^{(L)[d]}(\nu_1 - \nu_m) = \int \prod_{\ell=1}^{L} \frac{d^d k_\ell}{(2\pi)^d} \frac{1}{\prod_{j=1}^{m} [q_j^2 - m_j^2 + i\phi]^{\nu_j}} \quad ; \quad D_j = q_j^2 - m_j^2 + i\phi$$

WHERE WE CAN USE THE (GENERAL) SCHWINGER RAPRESENTATION:

$$\frac{1}{[q_j^2 - m_j^2]^{\nu_j}} = \frac{1}{\Gamma(\nu_j)} \int_0^\infty d\alpha_j \, \alpha_j^{\nu_j - 1} \, e^{i\alpha_j (q_j^2 - m_j^2)}$$

INDEX SUMMED

WE DIDN'T USE THE i, BUT IT'S ANOTHER DEFINITION (SEE ITZYKSON, ZUBER CAP 6.2.3)

THAT BRING US TO WRITE ITS EXPONENT AS:

$$\sum_j \alpha_j D_j = \sum_j \alpha_j (q_j^2 - m_j^2) = M_{\ell m} k_{\ell\mu} k_m^\mu + 2 Q_{e\mu} k_e^\mu + J$$

DROPPING $i\phi$

WHERE THE CONTRACTED INDEX (ℓ, m) ARE SUMMED AND GOES FROM 1 TO L. $\mathcal{M}$ IS A $L \times L$ MATRIX, LINEAR IN α_i; Q_ℓ ARE A LINEAR COMBINATION OF EXTERNAL MOMENTUM p_i AND α_i; AND J IS A SCALAR WHICH CONTAINS THE m_i^2 MASS TERMS AND THE SQUARE OF p_i^2.

WE CAN REWRITE THE LAST RELATION IN A MATRIX FORM:

$$\vec{k}^T \mathcal{M} \vec{k} + 2\vec{Q}^T \cdot \vec{k} + J$$

IN THIS WAY WE HAVE:

$$I_m^{(L)[d]} = \prod_{\ell=1}^{L} \int \frac{d^d k_\ell}{(2\pi)^d} \left(\prod_{i=1}^{m} \frac{1}{\Gamma(\nu_i)} \int_0^\infty d\alpha_i \; \alpha_i^{\nu_i - 1} \; e^{-\left(\vec{k}^T \mathcal{M} \vec{k} + 2\vec{Q}^T \cdot \vec{k} + J \right)} \right)$$

AND, AS WE KNOW FROM THE 2-LOOP INTEGRAL, WE HAVE TO COMPLETE THE SQUARE IN THE EXPONENTIAL. WE CAN PERFORM THE SHIFT;

$$\vec{k} \longrightarrow \vec{k}' = \vec{k} + \mathcal{M}^{-1} \vec{Q} \qquad \left(k'_\ell = k_\ell + \sum_{m=1}^{L} (\mathcal{M}^{-1})_{\ell m} Q_m \right)$$

SO WE HAVE:

$$k^T \mathcal{M} k + 2Q^T k + J \longrightarrow (k' - \mathcal{M}^{-1}Q)^T \mathcal{M} (k' - \mathcal{M}^{-1}Q) + 2Q^T (k' - \mathcal{M}^{-1}Q) + J$$

$$= \vec{k}'^T \mathcal{M} \vec{k}' - k'^T Q - Q^T k' + Q^T \mathcal{M}^{-1} Q + 2Q^T k' - 2Q^T \mathcal{M}^{-1} Q + J$$

$$= \vec{k}'^T \mathcal{M} \vec{k}' + J - \vec{Q}^T \mathcal{M}^{-1} \vec{Q}$$

AND THE INTEGRAL:

$$I_m^{(L)[d]} = \prod_{e=1}^{L} \int \frac{d^d k'_e}{(2\pi)^{d/2}} \left(\prod_{j=1}^{m} \frac{1}{\Gamma(\nu_j)} \int_0^\infty d\alpha_j \; \alpha_j^{\nu_j - 1} \exp\left\{ i \left(\vec{K}'^{\,T} M \vec{K}' + J - \vec{Q}^T M^{-1} \vec{Q} \right) \right\} \right)$$

SO WE CAN FACTORIZE THE EXPONENTIAL TO GET A WELL KNOWN GAUSSIAN INTEGRAL:

$$I_m^{(L)[d]} = \prod_{j=1}^{m} \frac{1}{\Gamma(\nu_j)} \int_0^\infty d\alpha_j \; \alpha_j^{\nu_j - 1} \; e^{-(J - i Q^T M^{-1} Q)} \underbrace{\int \prod_{e=1}^{L} \frac{d^d k'_e}{(2\pi)^{d/2}} \; e^{-K'^T M K'}}$$

IF WE REMEMBER:

$$\int_{-\infty}^{+\infty} dx_1 \dots dx_m \exp\{-x^T A x\} = \sqrt{\frac{\pi^m}{\det A}}$$

THIS IS:

$$\frac{1}{(4\pi)^{Ld/2}} \frac{i^L}{(\det M)^{d/2}}$$

SO

$$I_m^{(L)[d]} = \left(\frac{i}{(4\pi)^{d/2}} \right)^L \prod_{j=1}^{m} \frac{1}{\Gamma(\nu_j)} \int_0^\infty d\alpha_j \; \alpha_j^{\nu_j - 1} \; e^{-(J - Q^T M^{-1} Q)} \; \mu^{-d/2}$$

WHERE WE WROTE $\mu = \det M$ FIRST SYMANZIK POLYNOMIAL, WE CAN ALSO USE THE SECOND SYMANZIK POLYNOMIAL:

$$F = (J - Q^T M^{-1} Q)\, \mu$$

SO WE HAVE:

$$I_m^{(L)[d]} = \left(\frac{i}{(4\pi)^{d/2}}\right)^L \prod_{j=1}^{m} \frac{1}{\Gamma(\nu_j)} \int_0^\infty d\alpha_j\, \alpha_j^{\nu_j-1} \exp\left\{-\frac{F}{\mu}\right\} \mu^{-d/2}$$

WE CAN CHANGE VARIABLES FROM α_j (IN $[0,\infty)$) TO β_j (IN $[0,1]$) INSERTING THE IDENTITY:

$$1 = \int_0^\infty d\lambda\, \delta\left(\lambda - \sum_{j=1}^{m} \alpha_j\right)$$

AND DO $\alpha_j = \lambda \beta_j$, SO:

$$\prod_{j=1}^{m} d\alpha_j\, \delta\left(\lambda - \sum_{i=1}^{m} \alpha_i\right) = \prod_{j=1}^{m} d\beta_j\, \lambda^m\, \delta\left(\lambda - \lambda \sum_{i=1}^{m} \beta_i\right) d\lambda$$

$$= \lambda^m \prod_{j=1}^{m} d\beta_j\, d\lambda\, \frac{1}{\lambda}\delta\left(1 - \sum \beta\right) = \lambda^{m-1} \prod_{j=1}^{m} d\beta_j\, d\lambda\, \delta\left(1 - \sum_{i=1}^{m} \beta_i\right)$$

$$\prod_{j=1}^{m} \alpha_j^{\nu_j-1} = \prod \alpha_j^{\sum(\nu_j-1)} = \prod (\lambda\beta_i)^{\sum \nu_j - m} = \prod_{j=1}^{m} \beta_j^{\nu_j-1} \cdot \lambda^{\sum \nu_j - m}$$

AND BECAUSE μ IS AN HOMOGENEOUS FUNCTION, BY L-GRADE, IN α_j, AND F BY $(L+1)$-GRADE WE HAVE

$$\mu(\alpha) = \mu(\lambda\beta) = \lambda^L \mu(\beta)$$

$$F(\alpha) = F(\lambda\beta) = \lambda^{L+1} F(\beta)$$

$$\Rightarrow \frac{F}{\mu} = \lambda \frac{F}{\mu}$$

AND WE HAVE:

$$I_m^{(L)[d]} = \left(\frac{i}{(4\pi)^{d/2}}\right)^L \frac{1}{\prod_{j=1}^{m} \Gamma(\nu_j)} \int_0^1 \prod_{j=1}^{m} d\beta_j\, \beta_j^{\nu_j-1} \int_0^\infty d\lambda\, \lambda^{m-m+\sum \nu_j - 1}\, \delta\left(1 - \sum_{j=1}^{m} \beta_j\right) e^{-\lambda \frac{F}{\mu}} (\lambda^L \mu)^{-\frac{d}{2}}$$

(SUMMED; FOR THE HOMOGENEITY)

SO :

$$I_m^{(L)[d]} = \left(\frac{\dot{\pi}^{L}}{(4\pi)^{d/2}}\right)^L \frac{1}{\prod_{j=1}^{m}\Gamma(\nu_j)} \int_0^1 \prod_{j=1}^{m} d\beta_j\, \beta_j^{\nu_j-1} \int_0^\infty d\lambda\, \lambda^{\sum\nu_j - 1 - L\frac{d}{2}}\, \delta\left(1-\sum_{j=1}^{m}\beta_j\right) e^{-\lambda\frac{F}{\mu}}\, \mu^{-\frac{d}{2}}$$

$$= \left(\frac{\dot{\pi}^{L}}{(4\pi)^{d/2}}\right)^L \frac{1}{\prod_{j=1}^{m}\Gamma(\nu_j)} \int_0^1 \prod_{j=1}^{m} d\beta_j\, \beta_j^{\nu_j-1}\, \delta\left(1-\sum_{j=1}^{m}\beta_j\right) \underbrace{\int_0^\infty d\lambda\; \lambda^{\sum\nu_j - 1 - L\frac{d}{2}}\, e^{-\lambda\frac{F}{\mu}}}\; \mu^{-\frac{d}{2}}$$

WE CAN ANALYZE THE UNDERLINED TERM :

$$z = \lambda\frac{F}{\mu} \Rightarrow dz = d\lambda\frac{F}{\mu}$$

SO

$$\int_0^\infty d\lambda\, \lambda^{\sum\nu_j - 1 - L\frac{d}{2}}\, e^{-\lambda\frac{F}{\mu}} \Rightarrow \int_0^\infty dz \left(\frac{\mu}{F}\right)\cdot\left(\frac{\mu}{F}\right)^{\sum\nu_j - 1 - L\frac{d}{2}} e^{-z}\, z^{\sum\nu_j - L\frac{d}{2} - 1}$$

$$= \left(\frac{\mu}{F}\right)^{\sum\nu_j - L\frac{d}{2}} \int_0^\infty dz\; z^{\sum\nu_j - L\frac{d}{2} - 1}\, e^{-z}$$

$$= \left(\frac{\mu}{F}\right)^{\sum\nu_j - L\frac{d}{2}} \Gamma\left(\sum\nu_j - L\frac{d}{2}\right)$$

REMEMBER THAT THIS IS SUMMED

WE CONCLUDE :

$$I_m^{(L)[d]} = \left(\frac{\dot{\pi}^{L}}{(4\pi)^{d/2}}\right)^L \frac{\Gamma\left(\sum_{j=1}^{m}\nu_j - L\frac{d}{2}\right)}{\prod_{j=1}^{m}\Gamma(\nu_j)} \int_0^1 \prod_{j=1}^{m} d\beta_j\, \beta_j^{\nu_j-1}\, \delta\left(1-\sum_{j=1}^{m}\beta_j\right) \left(\frac{\mu^{\sum\nu_j - L\frac{d}{2}}}{F^{\sum\nu_j - L\frac{d}{2}}}\right) \mu^{-\frac{d}{2}} \quad (5)$$

so:

$$I_m^{(L)[d]} = \left(\frac{i}{(4\pi)^{d/2}}\right)^L \frac{\Gamma\left(\sum_{j=1}^m \nu_j - L\frac{d}{2}\right)}{\prod_{j=1}^m \Gamma(\nu_j)} \int_0^1 \prod_{j=1}^m d\beta_j \; \beta^{\nu_j - 1} \; \delta\left(1 - \sum_{j=1}^m \beta_j\right) \frac{\mu^{\sum \nu_j - (L+1)\frac{d}{2}}}{F^{\sum \nu_j - L\frac{d}{2}}} .$$

THE MINUS SIGN IS CONVENCTION (??), BUT EQ. 7 IN BOGNER, WEINZIERL'S ARTICLE FOLLOWS OUR NOTATION.