Adanced Quantum Field Theory

Exrcises

$\boxed{\text{Chapter 3}}$

EXERCISE 1

1 FROM THE DEFINITIONS :

$$\beta_\pm = \frac{1 \pm \beta}{2} \quad , \quad \beta = \sqrt{1 - \frac{4m^2}{p^2}}$$

SHOW THAT :

$$\beta_+ \beta_- = \frac{m^2}{p^2} .$$

2 USE THE TRANFORMATION $\alpha = y + \beta_+$ TO SHOW:

$$\hat{\Delta} = \frac{y(y+\beta)}{\beta_+ \beta_-} .$$

3 COMPLETE THE INTEGRATION OVER y TO SHOW:

$$\int_0^1 d\alpha \, \log(\hat{\Delta}) = -2 + \beta \log\left(-\frac{\beta_+}{\beta_-}\right)$$

SOLUTION 1 WE CAN EASILY SEE:

$$\beta_+ \beta_- = \frac{1+\beta}{2} \, \frac{1-\beta}{2} = \frac{1}{4}(1 - \beta^2) = \frac{1}{4}\left(1 - \left(1 - \frac{4m^2}{p^2}\right)\right) = \frac{m^2}{p^2} .$$

2 WE HAVE THE DEFINITION ! $\hat{\Delta} = 1 - \alpha(1-\alpha)\frac{p^2}{m^2}$

IF WE USE THE RESULT 1 AND PERFORM $\alpha \longrightarrow y + \beta_+$ WE GET

$$\hat{\Delta} = 1 - (y + \beta_+)(1 - y - \beta_+)\frac{p^2}{m^2}$$

EXPANDING :

$$\hat{\lambda} = 1 - \left(y - \frac{1+\beta}{2}\right)\left(1 - y - \frac{1+\beta}{2}\right)\frac{1}{\beta_+\beta_-}$$

$$= 1 - \frac{1}{4}(2y+1+\beta)(2-2y-1-\beta)\frac{1}{\beta_+\beta_-}$$

$$= 1 - \frac{1}{4}\frac{1}{\beta_+\beta_-}(2y+1+\beta)(-2y+1-\beta)$$

$$= 1 + \frac{1}{4}\frac{1}{\beta_+\beta_-}(2y+1+\beta)(2y-1+\beta)$$

$$= \frac{1}{4}\frac{1}{\beta_+\beta_-}\left[4\beta_+\beta_- + (2y+\beta)^2 - 1\right]$$

$$= \frac{1}{4}\frac{1}{\beta_+\beta_-}\left[4\,\frac{1-\beta^2}{4} + 4y^2 + 4y\beta + \beta^2 - 1\right]$$

$$= \frac{1}{4}\frac{1}{\beta_+\beta_-}\left[1 - \beta^2 + 4y(y+\beta) + \beta^2 - 1\right]$$

$$= \frac{1}{4}\frac{1}{\beta_+\beta_-}\left[4y(y+\beta)\right] = \frac{y(y+\beta)}{\beta_+\beta_-}.$$

3 THE TRANFORMATION $\alpha = y + \beta_+$ IMPLIES:

$$\int_0^1 d\alpha \longrightarrow \int_{-\beta_+}^{1-\beta_+} dy$$

$$\int_a^b dx \log x = \left[x \log x - x\right]_a^b$$

SO WE HAVE

$$\int_{-\beta_+}^{1-\beta_+} dy \log(\hat{\Delta}) = \int_{-\beta_+}^{1-\beta_+} dy \log\left(\frac{y(y+\beta)}{\beta_+\beta_-}\right)$$

$z = y + \beta$; $dy = dz$

$$\begin{cases} z_+ = 1 + \beta - \beta_+ \\ z_- = \beta - \beta_+ \end{cases}$$

$$= -\log(\beta_+\beta_-)\int_{-\beta_+}^{1-\beta_+} dy + \int_{-\beta_+}^{1-\beta_+} dy \log(y) + \int_{-\beta_+}^{1-\beta_+} dy \log(y+\beta)$$

$$= -\log(\beta_+\beta_-)\left[1 - \beta_+ + \beta_+\right] + \left[y \log y - y\right]_{-\beta_+}^{1-\beta_+} + \int_{\beta-\beta_+}^{1+\beta-\beta_+} dz \log z$$

$$= -\log(\beta_+\beta_-) + (1-\beta_+)\log(1-\beta_+) - (1-\beta_+) + \beta_+\log(-\beta_+) - \beta_+ +$$

$$+ \left[z \log z - z\right]_{\beta-\beta_+}^{1+\beta-\beta_+}$$

$$= -\log(\beta_+\beta_-) + (1-\beta_+)\log(1-\beta_+) + \beta_+\log(-\beta_+) - 1 +$$

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$$+ (1+\beta-\beta_+)\log(1+\beta-\beta_+) - (1+\beta-\beta_+) - (\beta-\beta_+)\log(\beta-\beta_+) + (\beta-\beta_+)$$

$$= -\log(\beta_+\beta_-) + \log(1-\beta_+) + \beta_+\log\left(-\frac{\beta_+}{1-\beta_+}\right) - 1 + \log(1+\beta-\beta_+) +$$
$$+ (\beta-\beta_+)\log\left(\frac{1+\beta-\beta_+}{\beta-\beta_+}\right) - 1$$

$$= \log\left(\frac{1-\beta_+}{\beta_+\beta_-}\right) + \beta_+\log\left(-\frac{\beta_+}{1-\beta_+}\right) - 2 + \log(1+\beta-\beta_+) + (\beta-\beta_+)\log\left(\frac{1+\beta-\beta_+}{\beta-\beta_+}\right)$$

WE CAN SEE:

$$1-\beta_+ = 1 - \frac{1+\beta}{2} = \frac{1}{2}(2-1-\beta) = \frac{1-\beta}{2} = \beta_-$$

SO:

$$= \log\left(\frac{\beta_-}{\beta_+\beta_-}\right) + \beta_+\log\left(-\frac{\beta_+}{\beta_-}\right) - 2 + \log(\beta+\beta_-) + (\beta-\beta_+)\log\left(\frac{\beta+\beta_-}{\beta-\beta_+}\right)$$

ALSO:

$$\beta+\beta_- = \beta + \frac{1-\beta}{2} = \frac{1}{2}(2\beta+1-\beta) = \frac{1+\beta}{2} = \beta_+$$

$$\beta-\beta_+ = \beta - \frac{1+\beta}{2} = \frac{1}{2}(2\beta-1-\beta) = -\frac{1-\beta}{2} = -\beta_-$$

SO WE HAVE:

$$= \log\left(\frac{1}{\beta_+}\right) + \beta_+\log\left(-\frac{\beta_+}{\beta_-}\right) - 2 + \log\beta_+ - \beta_-\log\left(-\frac{\beta_+}{\beta_-}\right)$$

$$= -2 + \log\left(\frac{\beta_+}{\beta_+}\right) + (\beta_+ - \beta_-)\log\left(-\frac{\beta_+}{\beta_-}\right)$$

$$= -2 + \frac{1}{2}\left[(1+\beta) - (1-\beta)\right]\log\left(-\frac{\beta_+}{\beta_-}\right)$$

$$= -2 + \beta\log\left(-\frac{\beta_+}{\beta_-}\right).$$